

Transfer function

$$T.F = \frac{L.T\{o/p\}}{L.T\{i/p\}} \bigg|_{\text{initial conditions} = 0}$$

$$T(s) = \frac{C(s)}{R(s)}$$

Mathematical Models of control systems:

differential equation

$$\boxed{\frac{d^2 y}{dt^2} + \frac{dy}{dt} + y(t) = x(t)} \checkmark$$

- 1) Mechanical Translational systems
- 2) Mechanical Rotational systems

1) Mechanical Translational systems:

- i) Mass. ✓
- ii) Spring. ✓
- iii) Dash pot element. ✓

Newton's second law of motion:-

Sum of applied forces = sum of opposite forces

List of symbols:

x = displacement, (m)

$v = \frac{dx}{dt}$ = velocity m/sec

$a = \frac{dv}{dt} = \frac{d^2x}{dt^2}$ m/sec²

f = applied force (N)

f_m = opposing force offered by 'm' (N)

f_B = " " ^{spring}
~~dashpot~~

f_c = " " ^{spring}

M = Mass in kg

K = stiffness of spring N/m

B = friction N-sec/m

Force Balance equations in ideal elements



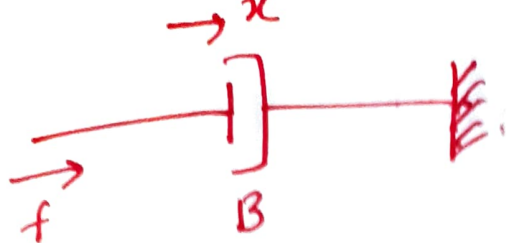
$$f_m \propto a \quad f_m \propto \frac{d^2x}{dt^2}$$

$$f_m = M \cdot \frac{d^2x}{dt^2}$$

By Newton's law $f = f_m = M \cdot \frac{d^2x}{dt^2}$



2)



$$f_B \propto \text{Velocity}$$

$$f_B \propto \frac{dx}{dt}$$

$$f_B = B \cdot \frac{dx}{dt}$$

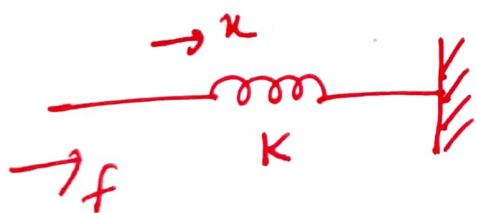
$$\boxed{f = f_B = B \cdot \frac{dx}{dt}} \rightarrow (2)$$



$$f_B \propto \frac{d}{dt}(x_1 - x_2)$$

$$f = f_B = B \frac{d}{dt}(x_1 - x_2) \rightarrow (3)$$

3)

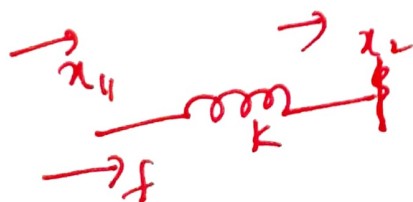


$$f_K \propto \text{displacement}$$

$$f_K \propto x$$

$$f_K = Kx$$

$$\boxed{f = f_K = Kx} \rightarrow (4)$$



$$\boxed{f = f_K = K(x_1 - x_2)} \rightarrow (5)$$

Steps to Determine T-F:

1) Identify no. of Nodes. where $\text{no. of nodes} = \text{N. of mass elements}$

2) Assume that linear displacements $x_1, x_2, \dots, x_n$

$$\frac{dx}{dt} = v, \quad \frac{d^2x}{dt^2} = a.$$

3) Draw the free body diagrams at the nodes.
by using Newton's law.

4) For each free body diagram, write one D.E.

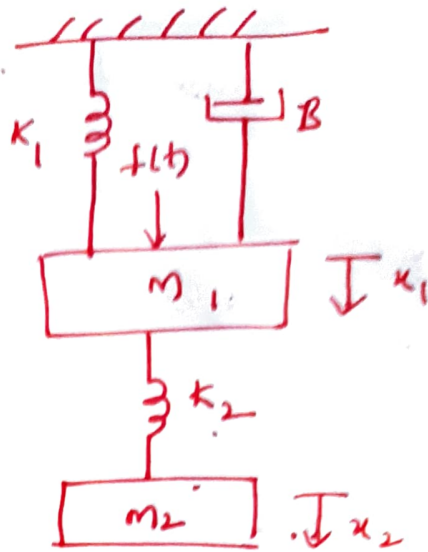
5) Use L.T to convert them into A.E
eliminate intermediate variable to find T-F

$$L\{x(t)\} = X(s)$$

$$L\left\{\frac{d}{dt}x(t)\right\} = s \cdot X(s)$$

$$L\left\{\frac{d^2}{dt^2}x(t)\right\} = s^2 X(s)$$

Def. the T.F $\frac{X_2(s)}{F(s)}$

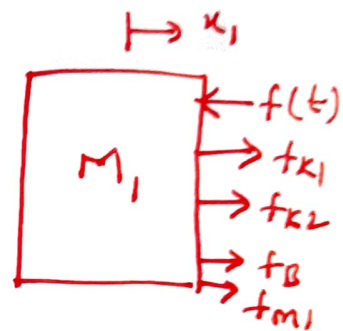


No. of Nodes are 2

Free body diagram at Node m_1

According to Newton's law

$$f(t) = f_{k1} + f_{k2} + f_B + f_{m1}$$



$$f(t) = K_1 x_1 + K_2 (x_1 - x_2) + B \frac{dx_1}{dt} + M_1 \frac{d^2 x_1}{dt^2}$$

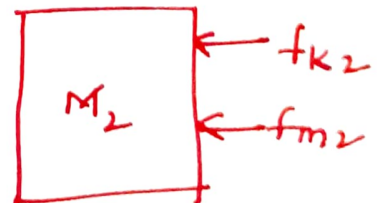
Apply . L.T

$$F(s) = K_1 X_1(s) + K_2 [X_1(s) - X_2(s)] + B s X_1(s) + M_1 s^2 X_1(s)$$

$$F(s) = [K_1 + K_2 + B s + M_1 s^2] X_1(s) - K_2 X_2(s) \quad \rightarrow (1)$$

Free body diagram at Node m_2

$$f_{k2} + f_{m2} = 0$$



$$K_2 (x_2 - x_1) + m_2 \frac{d^2 x_2}{dt^2} = 0$$

Apply L.T

$$K_2 [X_2(s) - X_1(s)] + M_2 s^2 X_2(s) = 0$$

$$X_2(s) = \frac{K_2}{K_2 + M_2 s^2} X_1(s) \quad \rightarrow (2)$$

sub ② in ①

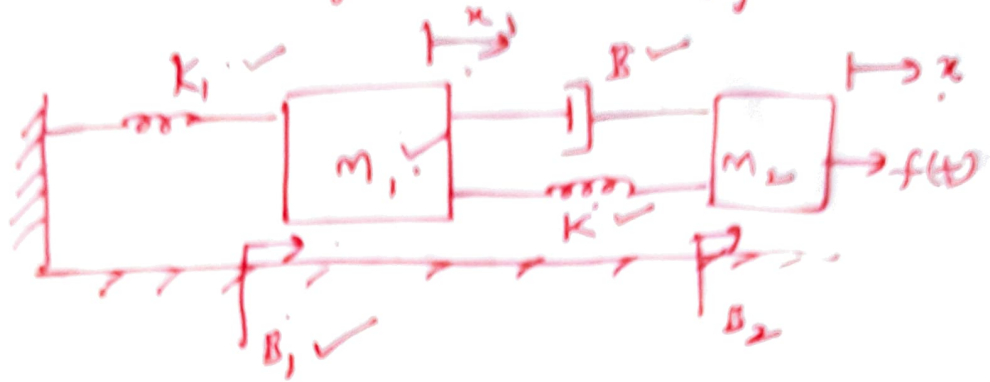
~~X₂(s)~~

$$F(s) = X_2(s) \left[\frac{m_2 s^2 + k_2}{k_2} \right] [m_1 s^2 + B s + K_1 + k_2] - X_2(s) k_2$$

$$\frac{X_2(s)}{F(s)} = \frac{k_2}{[m_1 s^2 + B s + K_1 + k_2] [m_2 s^2 + k_2] - k_2^2}$$

write differential equation for following

Mechanical translational system shown in fig & find T.F.

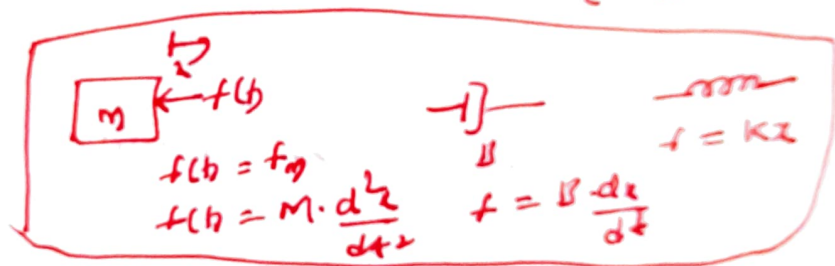


Sol:

$f(t) \rightarrow$ applied force

$x \rightarrow$ o/p displacement

$$\text{Req. T.F} = \frac{L\{f(t)\}}{L\{x(t)\}} = \frac{x(s)}{f(s)}$$

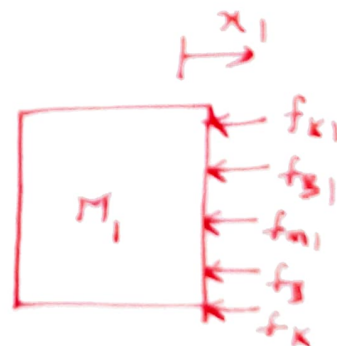


$$\text{No. of nodes} = \text{No. of Mass elements} = 2$$

Free body diagram at Node M_1

According to Newton's law

$$f_{K1} + f_{B1} + f_{m1} + f_B + f_K = 0$$



$$K_1 x_1 + B_1 \frac{dx_1}{dt} + m_1 \frac{d^2 x_1}{dt^2} + B \frac{d}{dt}(x_1 - x) + K(x_1 - x) = 0$$

L.T.

$$K_1 x_1(s) + B_1 s x_1(s) + m_1 s^2 x_1(s) + BS(x_1(s) - x(s)) + K(x_1(s) - x(s)) = 0$$

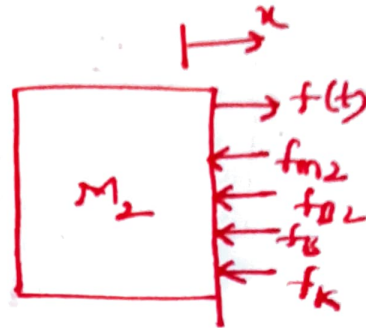
$$[m_1 s^2 + B_1 s + K_1 + BS + K] x_1(s) = [BS + K] x(s)$$

$$x(s) = ?$$

$$X_1(s) = \frac{(Bs + k) X(s)}{M_1 s^2 + (B_1 + B)s + (K_1 + K)} \rightarrow (1)$$

Free body diagram at Node M_2

$$f(t) = f_{m_2} + f_{B_2} + f_B + f_K$$



$$f(t) = M_2 \cdot \frac{d^2 x}{dt^2} + B_2 \cdot \frac{dx}{dt} + B \frac{d}{dt}(x - x_1) + K(x - x_1) \Rightarrow$$

Apply L.T.

$$F(s) = M_2 s^2 X(s) + B_2 s X(s) + B s (X(s) - X_1(s)) + K (X(s) - X_1(s))$$

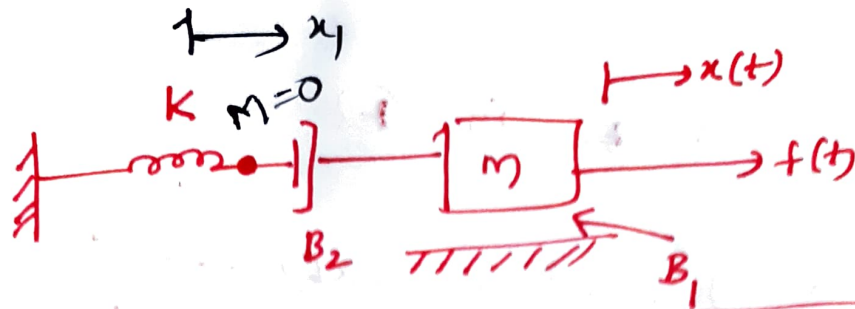
$$F(s) = [M_2 s^2 + (B_2 + B)s + K] X(s) - (Bs + K) X_1(s) \rightarrow (2)$$

sub (1) in (2)

$$F(s) = [M_2 s^2 + (B + B_2)s + K] X(s) - \frac{(Bs + K)(Bs + K) X(s)}{M_1 s^2 + (B_1 + B)s + K_1 + K}$$

$$\frac{X(s)}{F(s)} = \frac{M_1 s^2 + (B_1 + B)s + (K_1 + K)}{[M_1 s^2 + (B_1 + B)s + K_1 + K][M_2 s^2 + (B + B_2)s + K] - [Bs + K]^2}$$

③ →

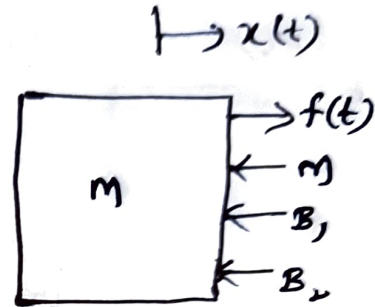


No. of Nodes = 2

$$T.F = \frac{X(s)}{F(s)}$$

Free body diagram at Node 'm':

$$f(t) = f_m + f_{B1} + f_{B2}$$



$$f(t) = m \cdot \frac{d^2 x}{dt^2} + B_1 \frac{dx}{dt} + B_2 \frac{d}{dt} (x - x_1)$$

Apply L.T.

$$F(s) = ms^2 x(s) + B_1 s x(s) + B_2 s (x(s) - x_1(s))$$

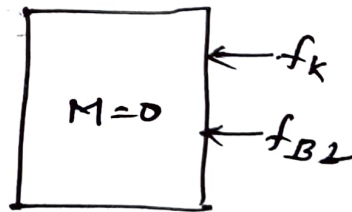
$$F(s) = [ms^2 + (B_1 + B_2)s] x(s) - B_2 s x_1(s)$$

→ ①

Free body diagram at Node (M=0)

$$f_{B2} + f_k = 0$$

$$B_2 \cdot \frac{d}{dt} [x_1 - x] + K x_1 = 0$$



L.T

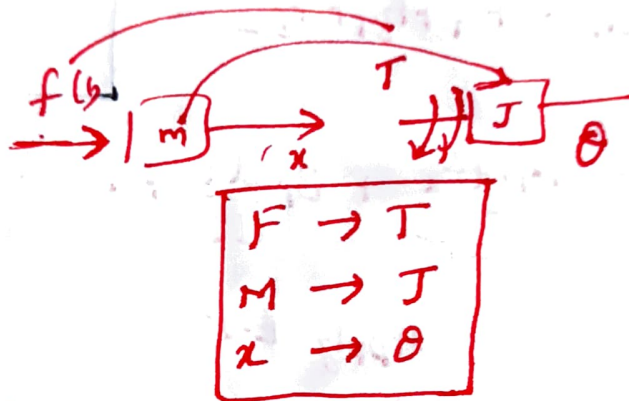
$$B_2 s [x_1(s) - x(s)] + K x_1(s) = 0$$

$$x_1(s) = \frac{B_2 s}{B_2 s + K} x(s) \rightarrow ②$$

sub ② in ①

$$F(s) = [ms^2 + (B_1 + B_2)s] X(s) - B_2 s \left(\frac{B_2 s}{B_2 s + K} \right) X(s)$$

$$\frac{X(s)}{F(s)} = \frac{B_2 s + K}{[ms^2 + (B_1 + B_2)s][B_2 s + K] - (B_2 s)^2}$$



Mechanical Rotational Systems:

- 1) Moment of Inertia $[J]$
- 2) Dash-pot element $[B]$
- 3) torsional spring $[K]$

$$\left\{ \begin{array}{l} \theta = \text{Angular displacement} \\ \frac{d\theta}{dt} = \text{Angular velocity} \\ \frac{d^2\theta}{dt^2} = \text{Angular acceleration} \\ T = \text{Torque} \\ J = \text{Moment of Inertia} \\ B = \text{Dash-pot coefficient} \\ K = \text{Torsional spring constant} \end{array} \right.$$

Torque Balanced equations of Ideal elements

1) Mass element

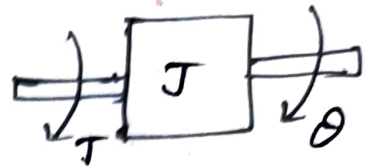
$$T = T_J$$

$$T_J \propto \frac{d^2\theta}{dt^2}$$

$$T_J = J \cdot \frac{d^2\theta}{dt^2}$$

According to Newton's law

$$T = J \cdot \frac{d^2\theta}{dt^2}$$



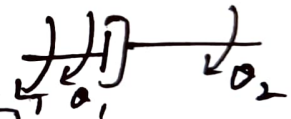
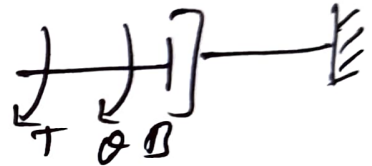
2) Dash-pot element

$$T_B \propto \frac{d\theta}{dt}$$

$$T_B = B \cdot \frac{d\theta}{dt}$$

$$T = T_B = B \cdot \frac{d\theta}{dt}$$

$$T = T_B = B \cdot \frac{d(\theta_1 - \theta_2)}{dt}$$



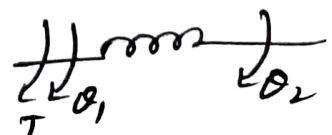
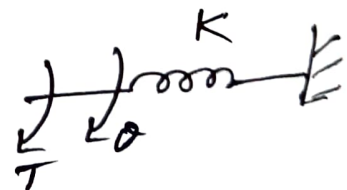
3) Spring (K)

$$T_K \propto \theta$$

$$T_K = K \cdot \theta$$

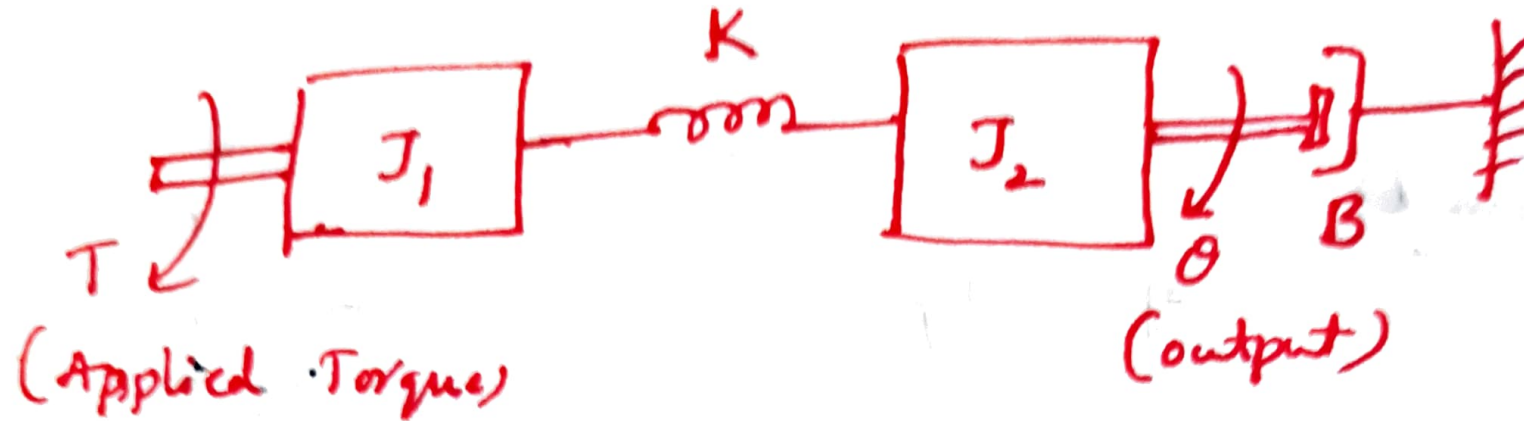
$$T = T_K = K \cdot \theta$$

$$T = T_K = K(\theta_1 - \theta_2)$$



Det. T.F. for the following rotational system.

④



Sol:

$$T.F = \frac{L\{T\}_{o/p}}{L\{T\}_{i/p}} = \frac{\theta(s)}{T(s)} \checkmark$$

Assignment

Electrical systems:

- i) resistor
- ii) Inductor
- iii) capacitor

Element

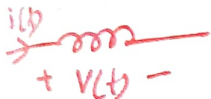
voltage

current



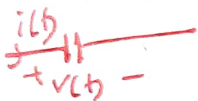
$$V(t) = R \cdot i(t)$$

$$i(t) = \frac{V(t)}{R}$$



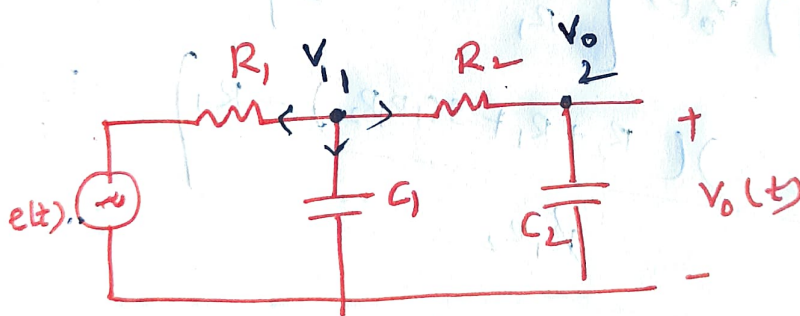
$$V(t) = L \cdot \frac{d i(t)}{dt}$$

$$i(t) = \int \frac{1}{L} V(t) dt$$



$$V(t) = \frac{1}{C} \int i(t) dt$$

$$i(t) = C \cdot \frac{dV(t)}{dt}$$



Sol:

$$T.F = \frac{L.T\{V_0(t)\}}{L.T\{e(t)\}} = \frac{V_0(s)}{E(s)}$$

Apply KCL at Node 1.

$$\frac{V_1 - e(t)}{R_1} + \frac{V_1 - V_0}{R_2} + C_1 \cdot \frac{dV_1}{dt} = 0$$

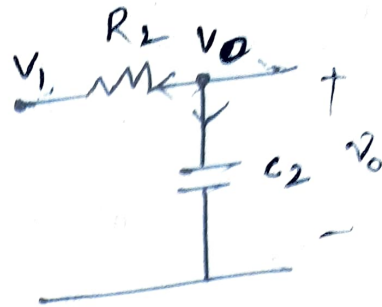
Apply L.T

$$\frac{V_1(s) - E(s)}{R_1} + \frac{V_1(s) - V_0(s)}{R_2} + C_1 s \cdot V_1(s) = 0$$

$$V_1(s) \left[\frac{1}{R_1} + sC_1 + \frac{1}{R_2} \right] - \frac{V_0(s)}{R_2} = \frac{E(s)}{R_1} \rightarrow (1)$$

At Node 2, Apply KCL

$$\frac{V_o(t) - V_1(t)}{R_2} + C_2 \cdot \frac{dV_o(t)}{dt} = 0$$



Apply LT

$$\frac{V_o(s)}{R_2} - \frac{V_1(s)}{R_2} + C_2 s \cdot V_o(s) = 0$$

$$V_1(s) = [1 + s C_2 R_2] V_o(s) \rightarrow (2)$$

sub. (2) in eq (1)

$$[1 + s R_2 C_2] V_o(s) \left[\frac{1}{R_1} + s C_1 + \frac{1}{R_2} \right] - \frac{V_o(s)}{R_2} = \frac{E(s)}{R_1}$$

$$V_o(s) \left[\frac{(1 + s R_2 C_2) (R_2 + R_1 + s C_1 R_1 R_2) - R_1}{R_1 R_2} \right] = \frac{E(s)}{R_1}$$

$$\boxed{\frac{V_o(s)}{E(s)} = \frac{R_2}{(1 + s R_2 C_2) (R_1 + R_2 + s C_1 R_1 R_2) - R_1}}$$

Electrical Analogues of Mechanical systems: Translational.

↓
equivalent

Mass } Analogous to { Resistor
Dash pot } Inductor
Spring } Capacitor

Input force (Mech) → voltage & current (elec)

output velocity → current & voltage

Two Types of analogies

i) Force - voltage analogy

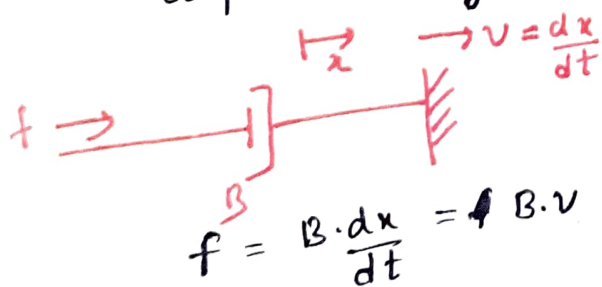
ii) Force - current analogy

Force - voltage Analogy:

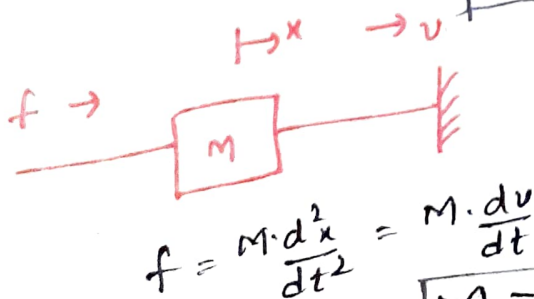
Mechanical system

Input: Force

output: Velocity



$B \rightarrow R$ (Resistor)



$M \rightarrow L$

Electrical system

Input: voltage source

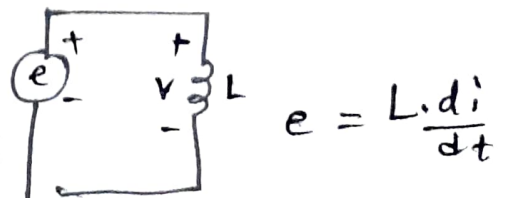
output: current through the element

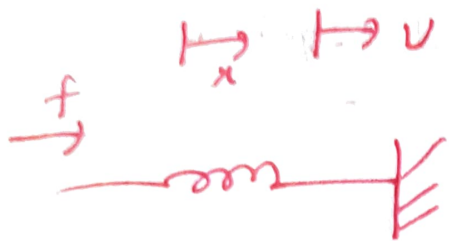
$$f = Bv$$

$$V = [R] I$$

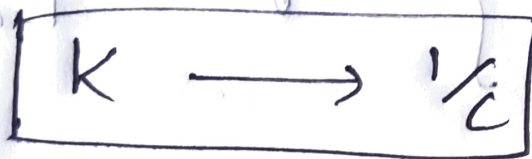


$$e = [L] \frac{di}{dt} \quad \text{Inductor}$$

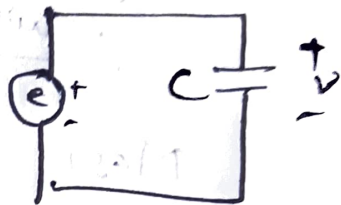




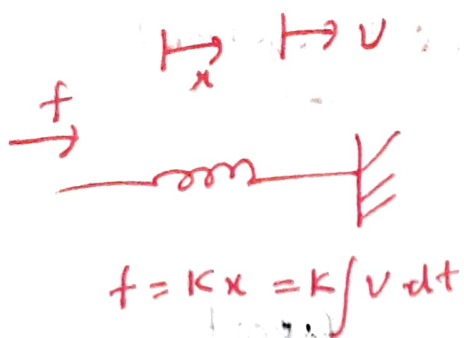
$$F = Kx = K \int v dt$$



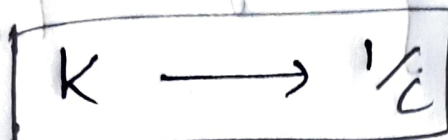
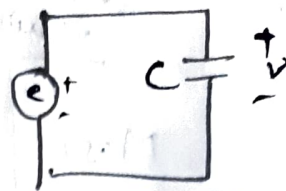
$$e = \left[\frac{1}{C} \right] \int v i dt$$



$$e = \frac{1}{C} \int i dt$$



$$e = \left[\frac{1}{C} \right] \int v i dt$$

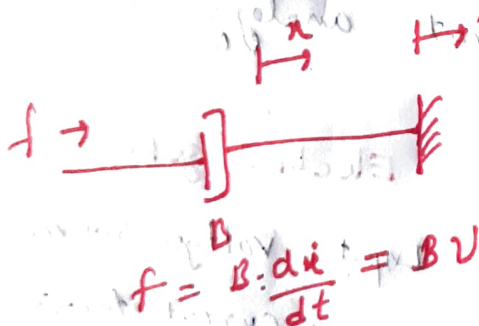


Force-current Analogy:-

Mechanical system

Input : Force

output : Velocity

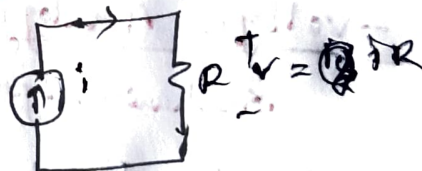


Input : Current source (i)

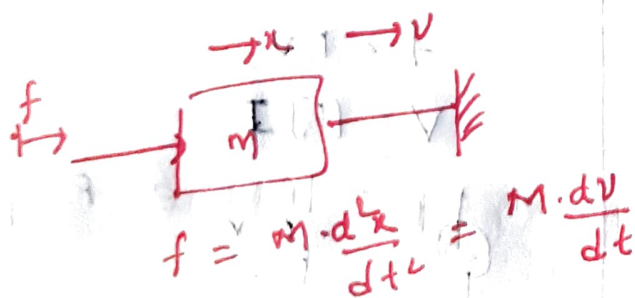
output : Voltage (v)

$$f = Bv$$

$$i = \left[\frac{1}{R} \right] v$$



$$B \longrightarrow \frac{1}{R}$$

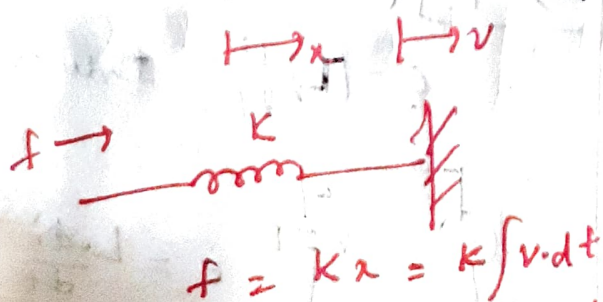


$$f = m \frac{dv}{dt}$$

$$i = \left[C \right] \frac{dv}{dt}$$



$$m \longrightarrow C$$



$$f = K \int v dt$$

$$i = \left[\frac{1}{L} \right] \int v dt$$



$$K \longrightarrow \frac{1}{L}$$

Mechanical

Translational system

F-V

$$B \rightarrow R$$

$$M \rightarrow L$$

$$K \rightarrow \frac{1}{C}$$

$$F \rightarrow V$$

$$\frac{dx}{dt} = v \rightarrow i$$

F-I

$$B \rightarrow \frac{1}{R}$$

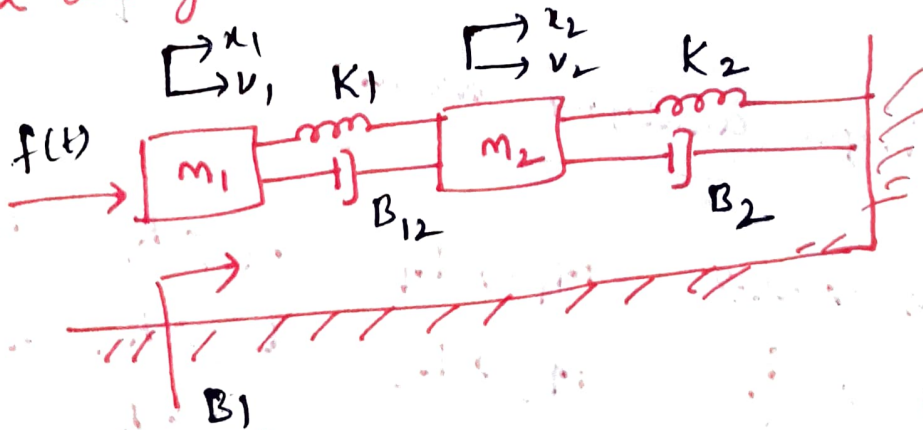
$$M \rightarrow C$$

$$K \rightarrow \frac{1}{L}$$

$$F \rightarrow I$$

$$\frac{dx}{dt} = v \rightarrow v$$

⑥ Write diff. equation and draw the F-V & F-I electrical analogous circuit for the following system



Sol:

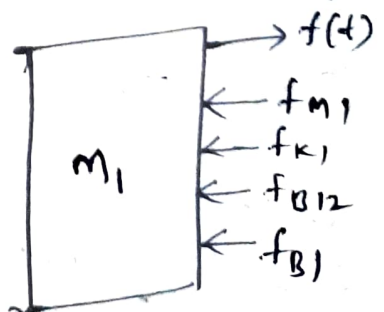
Nodes = 2

Free body diagram at Node m1

$$f(t) = f_{m1} + f_{K1} + f_{B12} + f_{B1}$$

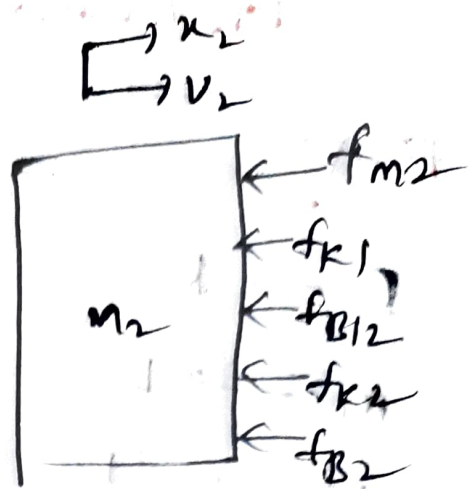
$$f(t) = m_1 \frac{d^2 x_1}{dt^2} + K_1 (x_1 - x_2) + B_{12} \frac{d}{dt} (x_1 - x_2) + B_1 \frac{dx_1}{dt}$$

$$f(t) = m_1 \frac{dv_1}{dt} + K_1 \int (v_1 - v_2) dt + B_{12} (v_1 - v_2) + B_1 v_1 \rightarrow \textcircled{1}$$



free body diagram at Node M_2

$$f_{m2} + f_{k1} + f_{B12} + f_{k2} + f_{B2} = 0$$



$$M_2 \frac{d^2 x_2}{dt^2} + K_1(x_2 - x_1) + B_{12} \frac{d}{dt}(x_2 - x_1) + K_2 x_2 + B_2 \frac{dx_2}{dt} = 0$$

$$\boxed{v = \frac{dx}{dt}}$$

$$M_2 \cdot \frac{dv_2}{dt} + K_1 \int (v_2 - v_1) dt + B_{12} (v_2 - v_1) + K_2 \int v_2 dt + B_2 v_2 = 0$$

→ ②

Force-voltage analogous circuit

$$f(t) \rightarrow e(t)$$

$$M_1 \rightarrow L_1$$

$$B_1 \rightarrow R_1 \quad K_1 \rightarrow \frac{1}{C_1}$$

$$v_1 \rightarrow i_1$$

$$M_2 \rightarrow L_2$$

$$B_2 \rightarrow R_2 \quad K_2 \rightarrow \frac{1}{C_2}$$

$$v_2 \rightarrow i_2$$

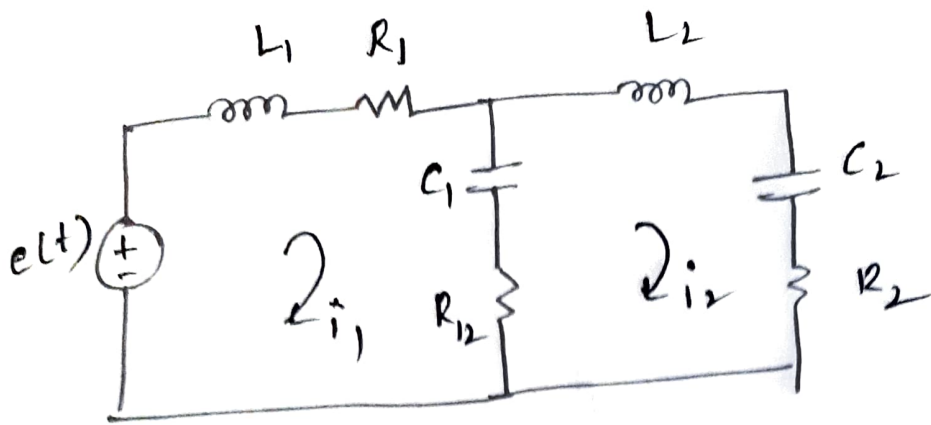
$$B_{12} \rightarrow R_{12}$$

$$\textcircled{1} \Rightarrow e(t) = \frac{L_1}{1} \cdot \frac{di_1}{dt} + \frac{1}{C_1} \int (i_1 - i_2) dt + R_{12} (i_1 - i_2) + R_1 i_1$$

→ ③

$$\textcircled{2} \Rightarrow L_2 \cdot \frac{di_2}{dt} + \frac{1}{C_2} \int i_2 dt + R_2 i_2 + \frac{1}{C_1} \int (i_2 - i_1) dt + R_{12} (i_2 - i_1) = 0$$

→ ④



F-V analogy circuit.

Force-current Analogous circuit:

$$f(t) \rightarrow i(t)$$

$$v_1 \rightarrow V_1$$

$$v_2 \rightarrow V_2$$

$$M_1 \rightarrow C_1$$

$$M_2 \rightarrow C_2$$

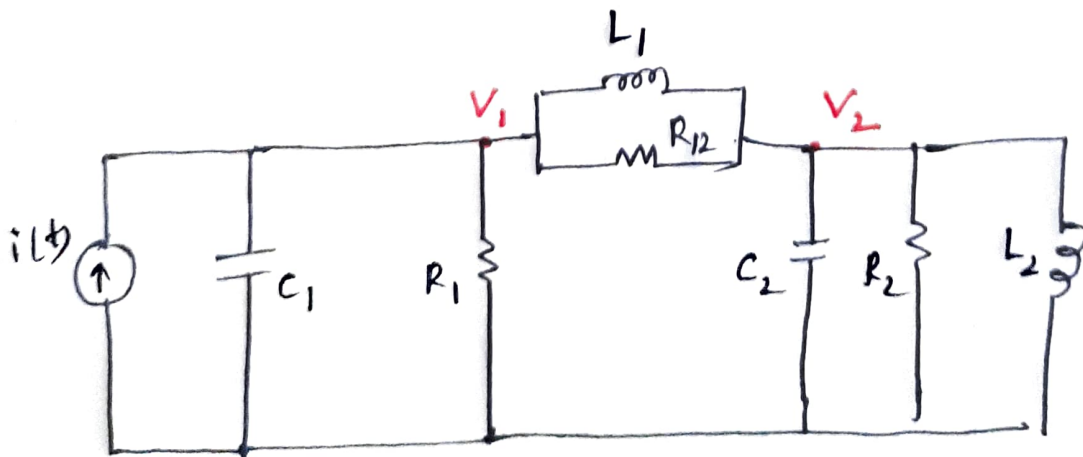
$$B_{12} \rightarrow \frac{1}{R_{12}} \quad K_1 \rightarrow \frac{1}{L_1}$$

$$B_1 \rightarrow \frac{1}{R_1} \quad K_2 \rightarrow \frac{1}{L_2}$$

$$B_2 \rightarrow \frac{1}{R_2}$$

$$\textcircled{1} \Rightarrow \left\{ \begin{aligned} C_1 \frac{dv_1}{dt} + \frac{1}{R_1} v_1 + \frac{1}{R_{12}} (v_1 - v_2) + \frac{1}{L_1} \int (v_1 - v_2) dt &= i(t) \end{aligned} \right. \rightarrow \textcircled{5}$$

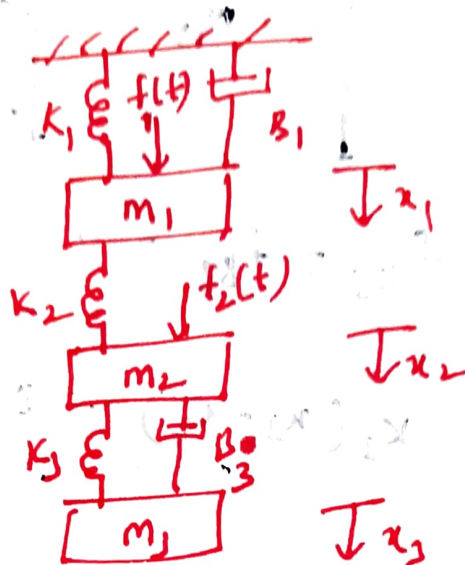
$$\textcircled{2} \Rightarrow \left\{ \begin{aligned} C_2 \frac{dv_2}{dt} + \frac{1}{R_2} v_2 + \frac{1}{L_2} \int v_2 dt + \frac{1}{R_{12}} (v_2 - v_1) + \frac{1}{L_1} \int (v_2 - v_1) dt &= 0 \end{aligned} \right. \rightarrow \textcircled{6}$$



F-C analogy circuit.

15-03-21

→ write D.E and find F-V, F-C analogy
follows
circuit for the system

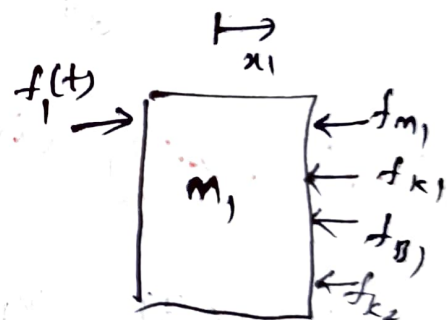


No. of Nodes = 3

Sol:

At Node m_1

$$f_1(t) = f_{m_1} + f_{k_1} + f_{B_1} + f_{k_2}$$



$$f_1(t) = m_1 \frac{d^2 x_1}{dt^2} + k_1 x_1 + B_1 \frac{dx_1}{dt} + k_2 (x_1 - x_2)$$

$$f_1(t) = m_1 \frac{dv_1}{dt} + k_1 \int v_1 dt + B_1 v_1 + k_2 \int (v_1 - v_2) dt \rightarrow (1)$$

At Node m_2

$$f_2(t) = f_{m_2} + f_{k_2} + f_{k_3} + f_{B_3}$$

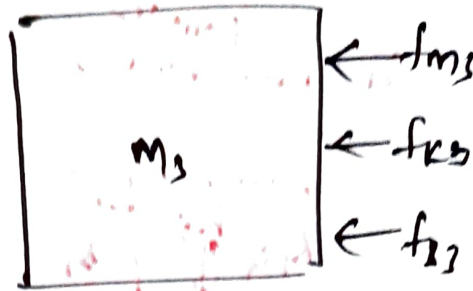


$$f_2(t) = m_2 \frac{d^2 x_2}{dt^2} + k_2 (x_2 - x_1) + k_3 (x_2 - x_3) + B_3 \frac{dx_2}{dt} (x_2 - x_3)$$

$$f_2(t) = m_2 \frac{dv_2}{dt} + k_2 \int (v_2 - v_1) dt + k_3 \int (v_2 - v_3) dt + B_3 (v_2 - v_3)$$

$\rightarrow (2)$

At Node m_3



$$f_{m3} + f_{k3} + f_{B3} = 0$$

$$m_3 \cdot \frac{d^2 x_3}{dt^2} + k_3 (x_3 - x_2) + B_3 \frac{d}{dt} (x_3 - x_2) = 0$$

$$m_3 \cdot \frac{dv_3}{dt} + k_3 \int (v_3 - v_2) dt + B_3 (v_3 - v_2) = 0 \quad \rightarrow (4)$$

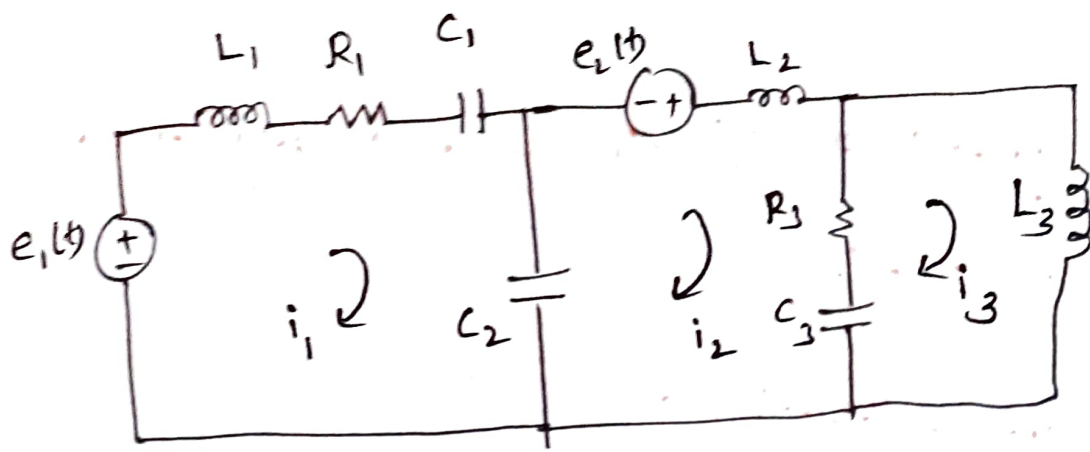
F-V analogy circuit:

$f_1(t) \rightarrow e_1(t)$	$v_1 \rightarrow i_1$	$m_1 \rightarrow L_1$	$B_1 \rightarrow R_1$	$k_1 \rightarrow \frac{1}{C_1}$
	$v_2 \rightarrow i_2$	$m_2 \rightarrow L_2$	$B_2 \rightarrow R_2$	$k_2 \rightarrow \frac{1}{C_2}$
$f_2(t) \rightarrow e_2(t)$	$v_3 \rightarrow i_3$	$m_3 \rightarrow L_3$		$k_3 \rightarrow \frac{1}{C_3}$

$$(1) \Rightarrow \underline{L_1} \cdot \underline{\frac{di_1}{dt}} + \underline{R_1} i_1 + \underline{\frac{1}{C_1}} \int i_1 dt + \underline{\frac{1}{C_2}} \int (i_1 - i_2) dt = e_1(t) \quad \rightarrow (4)$$

$$(2) \Rightarrow \underline{L_2} \cdot \underline{\frac{di_2}{dt}} + R_2 (i_2 - i_3) + \underline{\frac{1}{C_3}} \int (i_2 - i_3) dt + \underline{\frac{1}{C_2}} \int (i_2 - i_1) dt = e_2(t) \quad \rightarrow (5)$$

$$(3) \Rightarrow L_3 \cdot \frac{di_3}{dt} + \frac{1}{C_3} \int (i_3 - i_2) dt + R_3 (i_3 - i_2) = 0 \quad \rightarrow (6)$$



F-v analogy circuit.

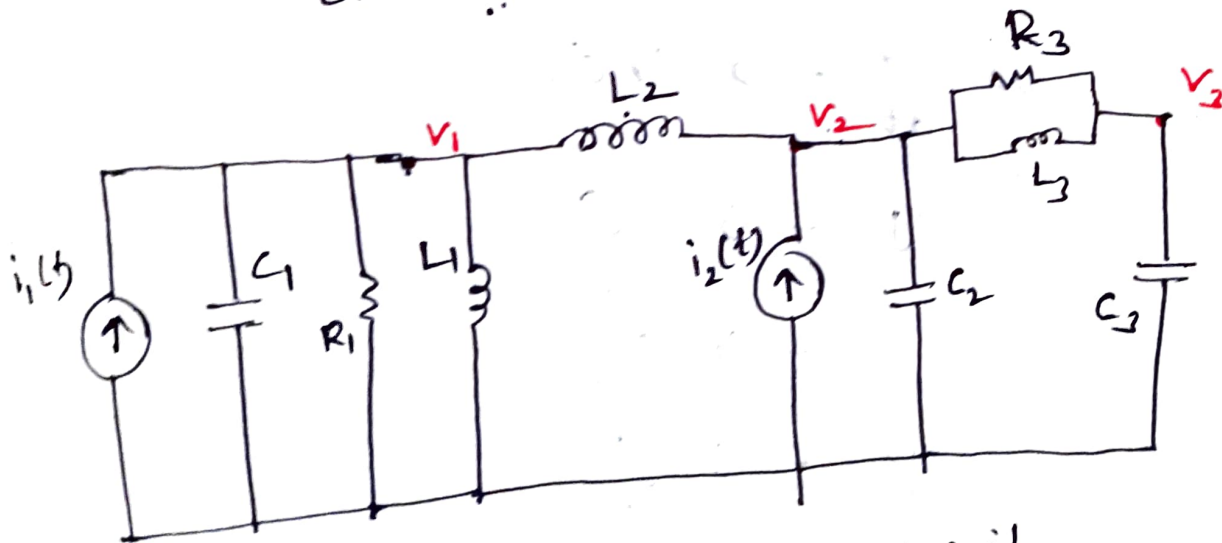
F-C analogous circuit.

$$\begin{array}{llll}
 f_1(t) \rightarrow i_1(t) & v_1 \rightarrow v_1 & m_1 \rightarrow C_1 & B_1 \rightarrow 1/R_1 \quad K_1 \Rightarrow 1/L_1 \\
 f_2(t) \rightarrow i_2(t) & v_2 \rightarrow v_2 & m_2 \rightarrow C_2 & B_2 \rightarrow 1/R_3 \quad K_2 \Rightarrow 1/L_2 \\
 & v_3 \rightarrow v_3 & m_3 \rightarrow C_3 & K_3 \Rightarrow 1/L_3
 \end{array}$$

$$① \Rightarrow i_1(t) = C_1 \frac{dv_1}{dt} + \frac{1}{R_1} v_1 + \frac{1}{L_1} \int v_1 dt + \frac{1}{L_2} \int (v_1 - v_2) dt$$

$$② \Rightarrow i_2(t) = C_2 \frac{dv_2}{dt} + \frac{1}{R_3} (v_2 - v_3) + \frac{1}{L_2} \int (v_2 - v_1) dt + \frac{1}{L_3} \int (v_2 - v_3) dt$$

$$③ \Rightarrow C_3 \frac{dv_3}{dt} + \frac{1}{R_3} (v_3 - v_2) + \frac{1}{L_3} \int (v_3 - v_2) dt = 0$$



F-C analogy circuit

Electrical Analogues of Mechanical Rotational Systems:

Torque-voltage analogy:

$$T \longrightarrow V(t)$$

$$\omega = \frac{d\theta}{dt} \longrightarrow i(t)$$

$$J \longrightarrow L$$

$$B \longrightarrow R$$

$$K \longrightarrow \frac{1}{C}$$

Torque-Current Analogy:

$$T \longrightarrow i(t)$$

$$\omega = \frac{d\theta}{dt} \longrightarrow V(t)$$

$$J \longrightarrow C$$

$$B \longrightarrow \frac{1}{R}$$

$$K \longrightarrow \frac{1}{L}$$

Block Diagrams :-

It is a pictorial representation of function performed by each component by using the flow of signal. The main elements of block diagram are

- i) Block
- ii) Branch Point / Take off point
- iii) summing point.

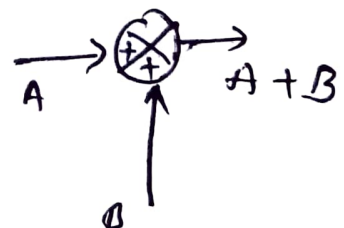
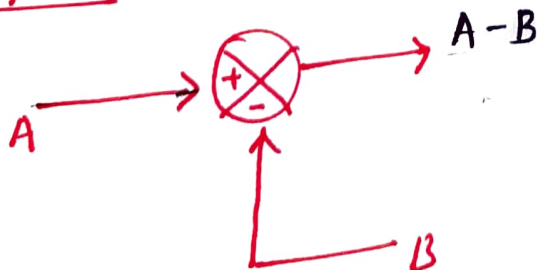
Block :



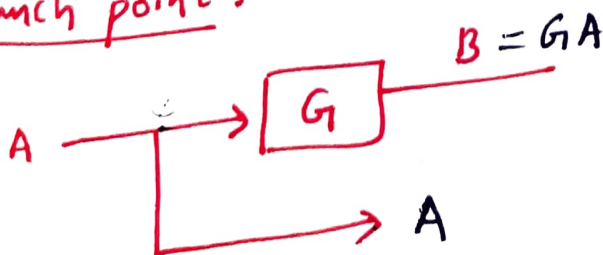
$$G(s) = \frac{B}{A} \Rightarrow B = A \cdot G(s)$$



Summing point:

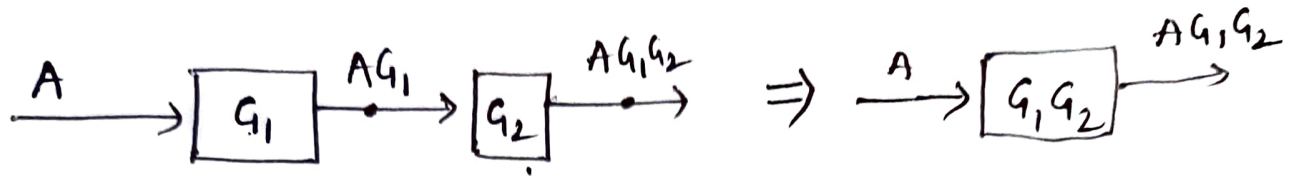


Branch point :

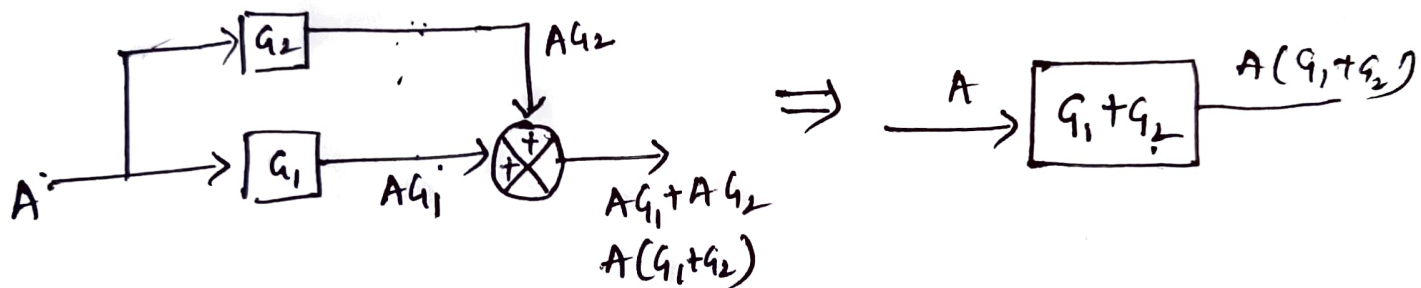


Rules of Block diagram Reduction:

Rule 1 :- combining blocks in cascade

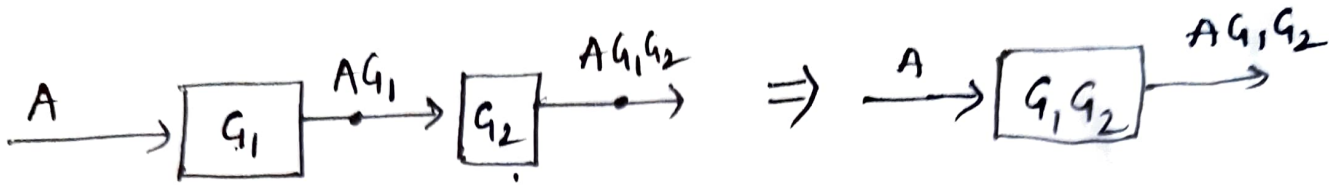


Rule 2 :- combining parallel blocks / forward paths

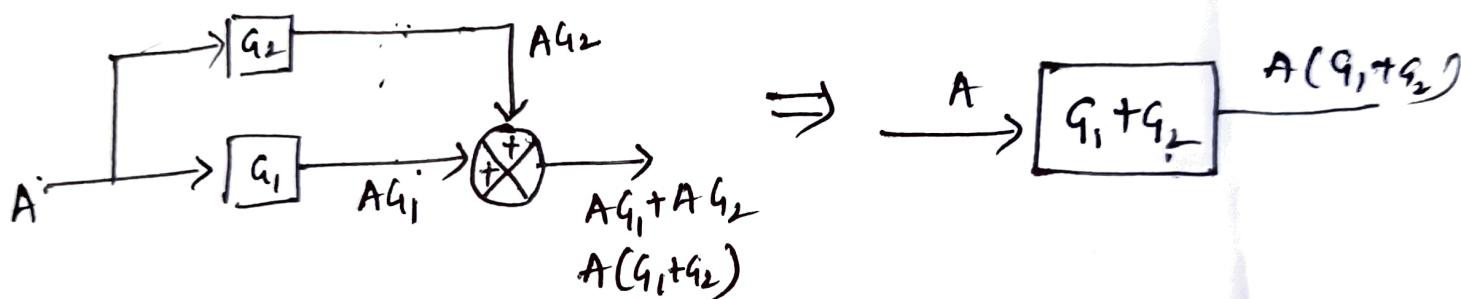


Rules of Block diagram Reduction:

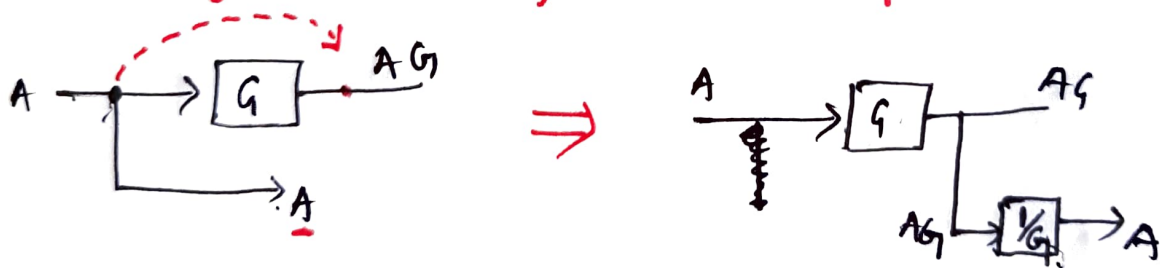
Rule 1:- combining blocks in cascade



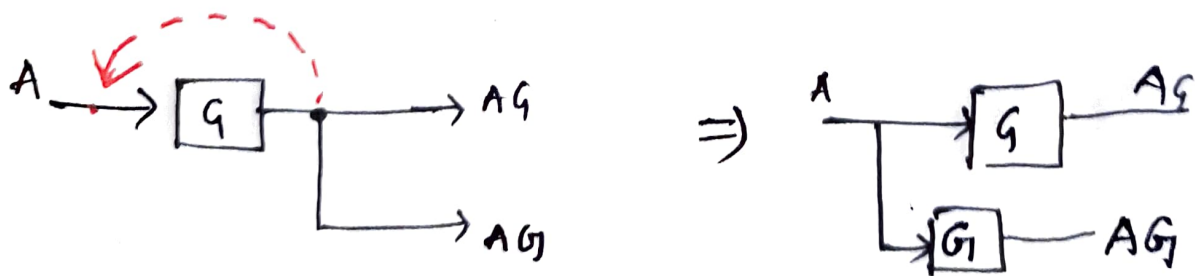
Rule 2:- combining parallel blocks / forward paths



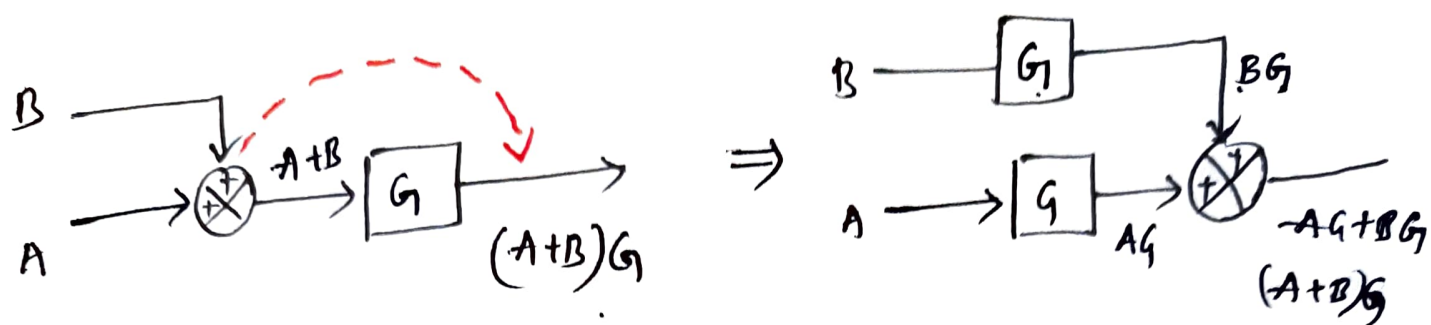
Rule 3:- Moving the branch point ahead of the block



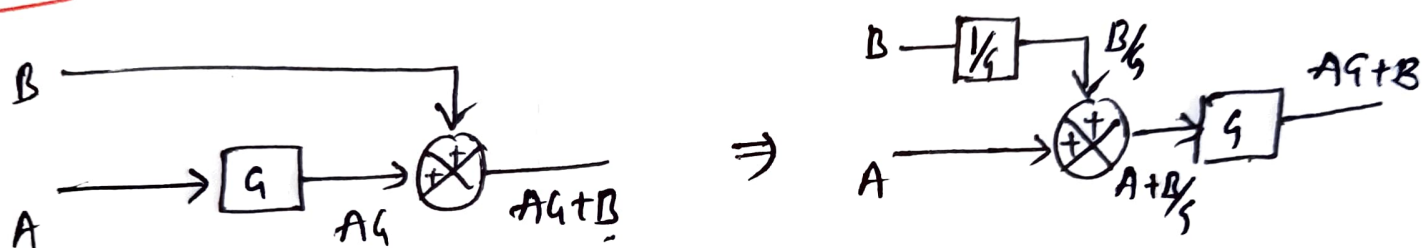
Rule 4:- Moving the branch point before the block



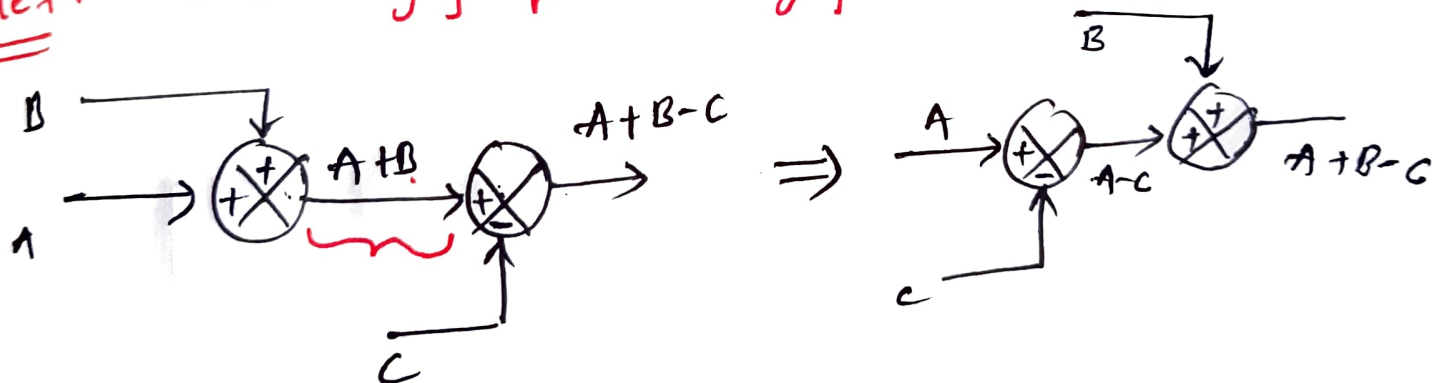
Rule 5: Moving the summing point ahead of the block.



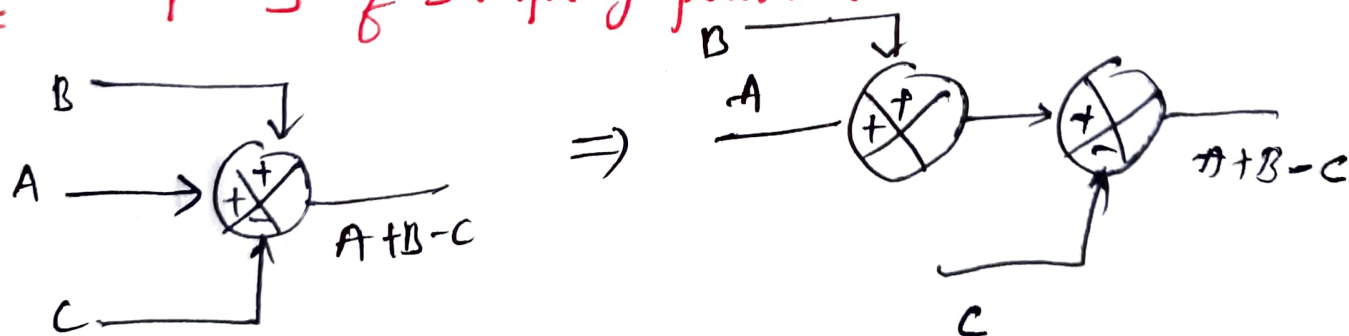
Rule 6: Moving the summing point before the block.



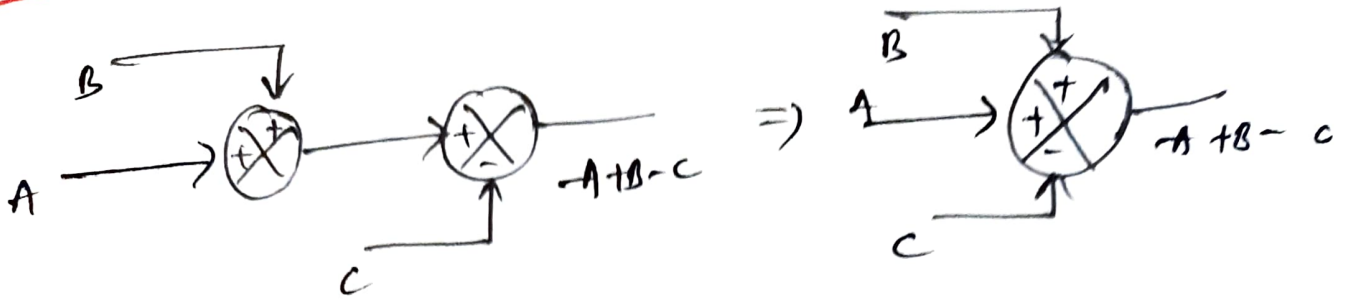
Rule 7: Interchanging 2 summing points.



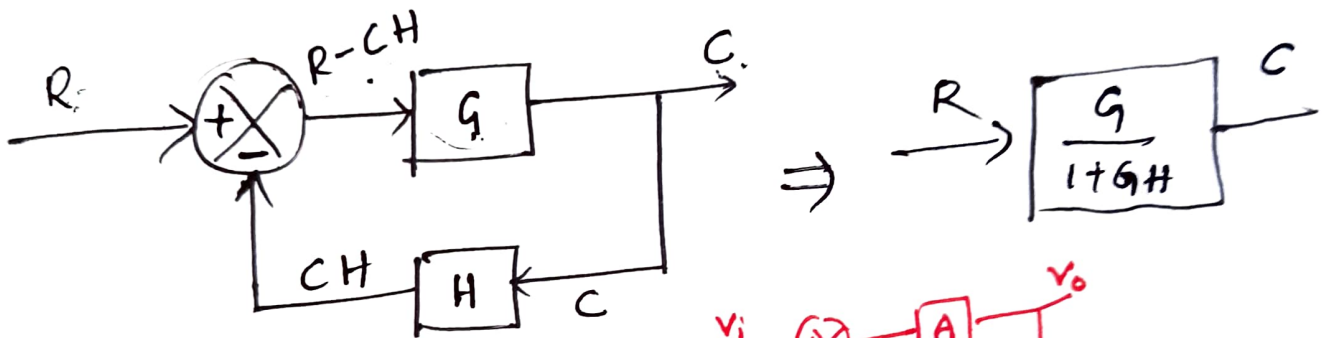
Rule 8: Splitting of summing point.



Rule 9: Combining summing points



Rule 10 Elimination of negative feedback loop

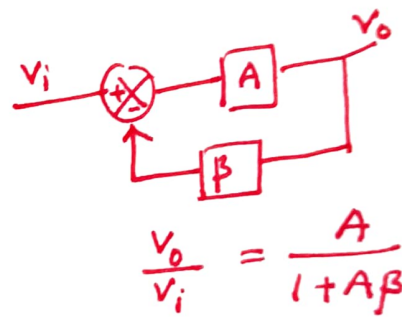


$$C = (R - CH)G$$

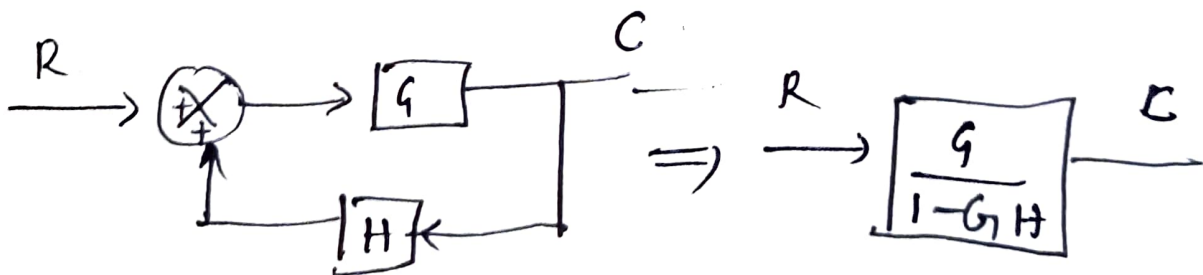
$$C = RG - CHG$$

$$C + CHG = RG$$

$$C(1 + GH) = GR \Rightarrow \frac{C}{R} = \frac{G}{1 + GH}$$

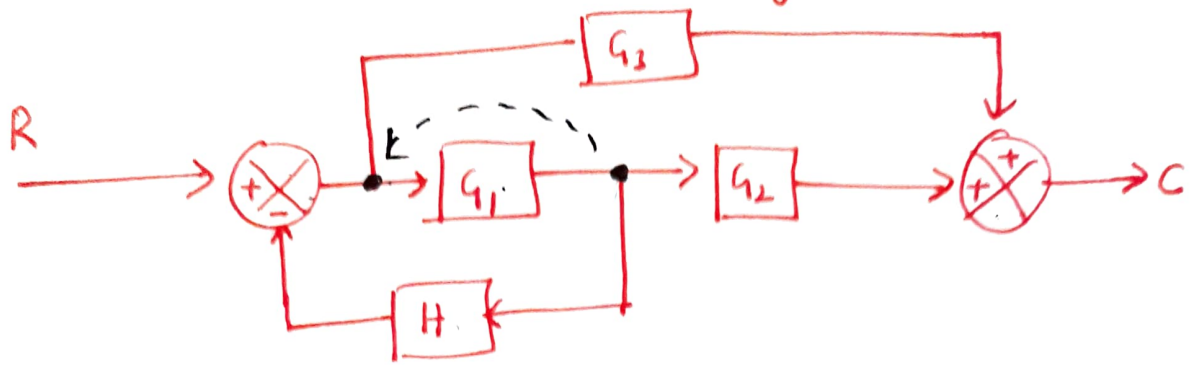


Rule 11) Elimination of positive feedback



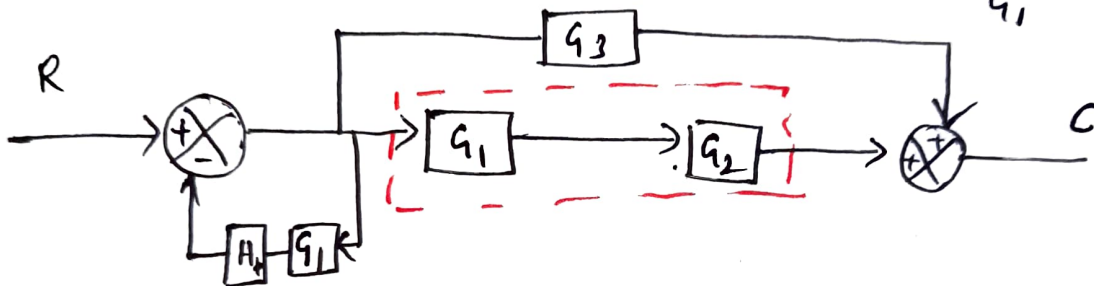
$$\boxed{\frac{C}{R} = \frac{G}{1 - GH}}$$

P1) Reduce the block diagrams and find C/R shown in fig.

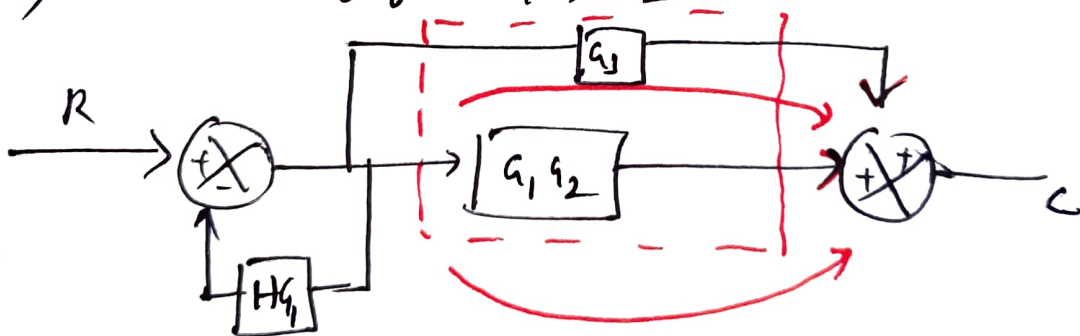


Sol:-

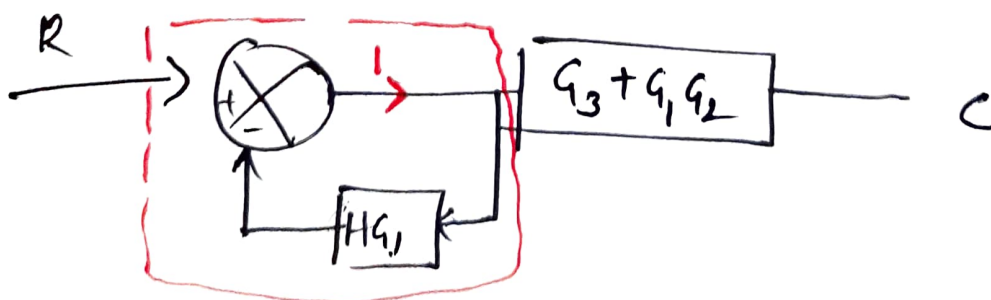
1) Moving of branchpoint before the block G_1



2) Cascading of G_1 & G_2



3) Combining parallel blocks

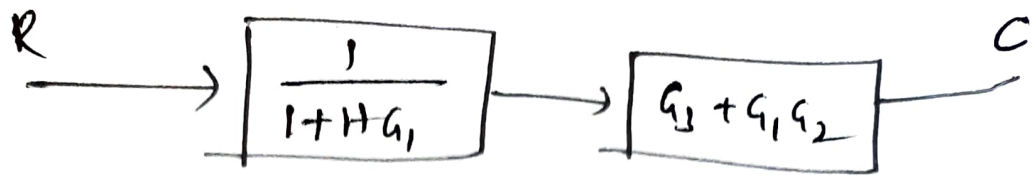


4) eliminate negative feedback loop

$$\begin{matrix} G = 1 \\ H = HG_1 \end{matrix}$$

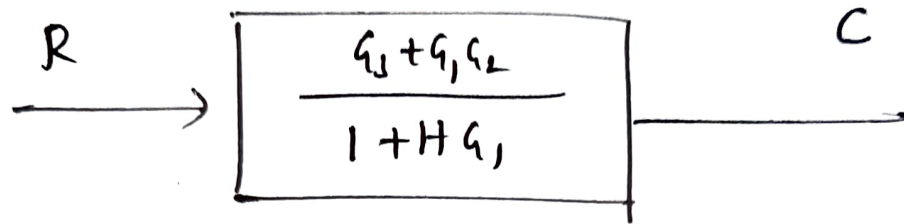
$$\frac{G}{1 + GH} \Rightarrow \frac{1}{1 + HG_1}$$

4)



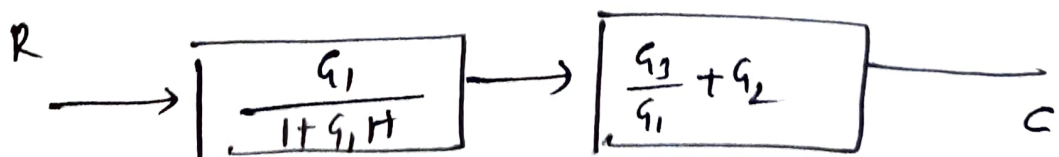
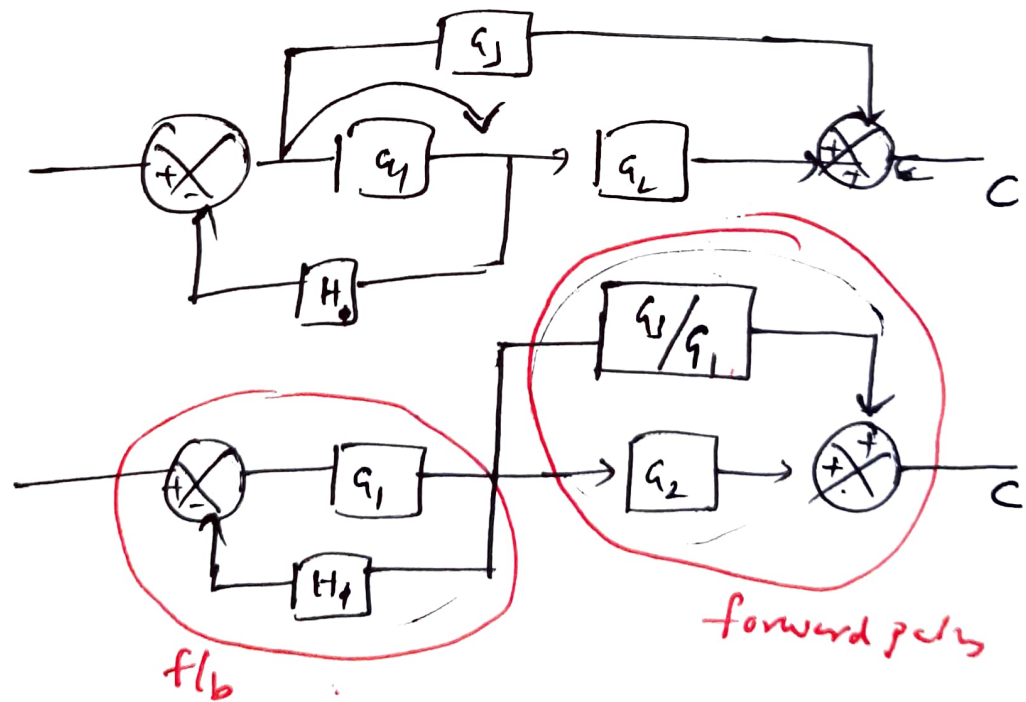
5)

combining 2 cascaded blocks



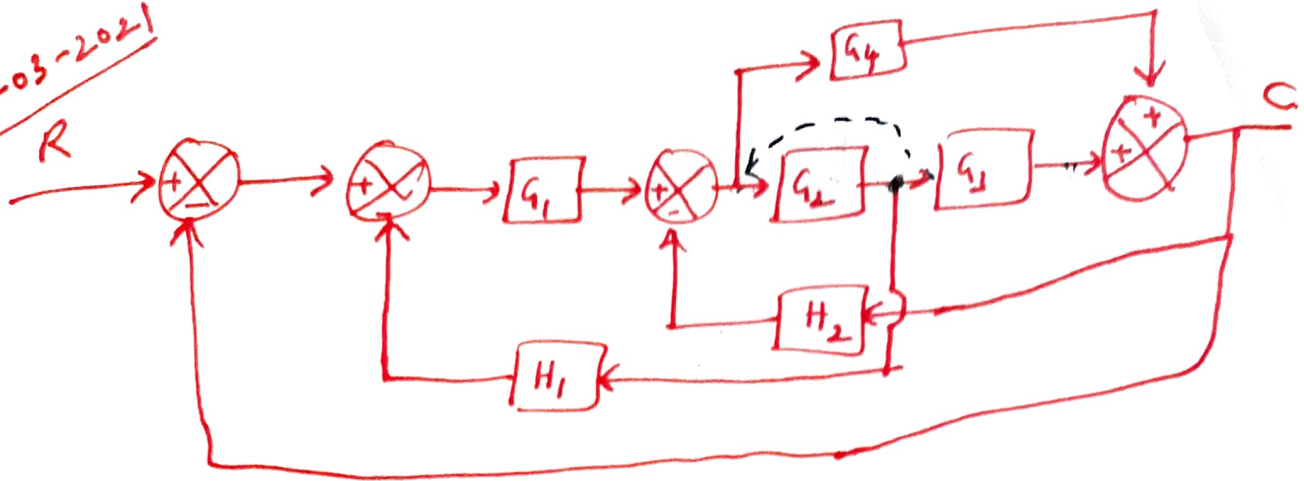
$$\frac{C}{R} = \frac{G_3 + G_1 G_2}{1 + HG_1}$$

Method-II

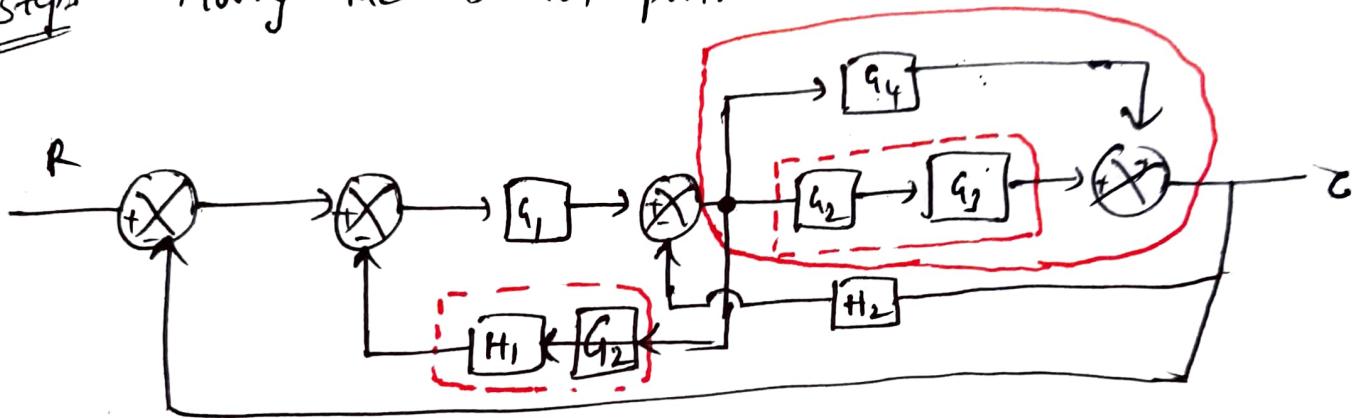


$$\frac{C}{R} = \frac{\left[\frac{G_3}{G_1} + G_2 \right] G_1}{1 + G_1 H} = \frac{G_1 G_2 + G_3}{1 + G_1 H}$$

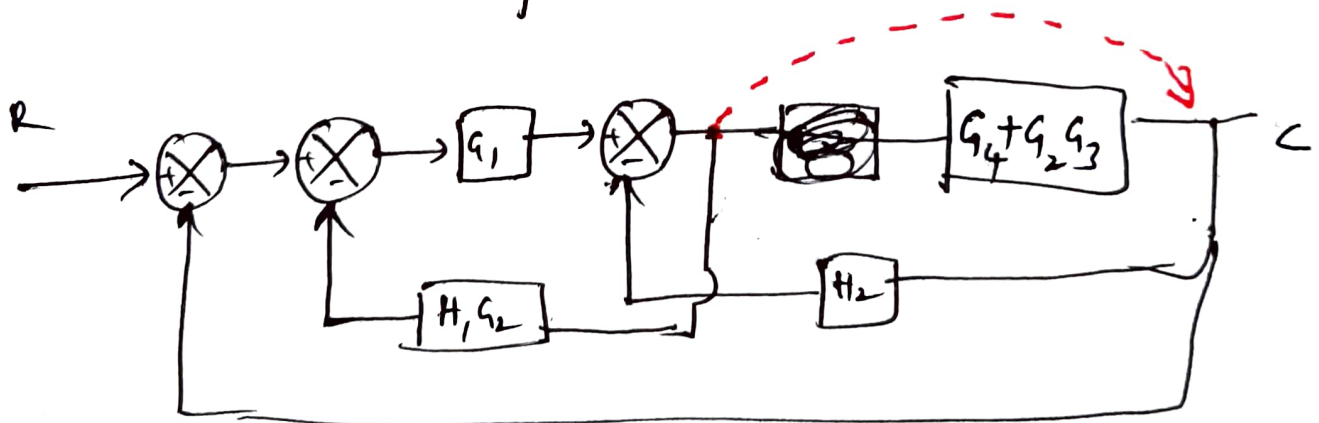
18-03-2021



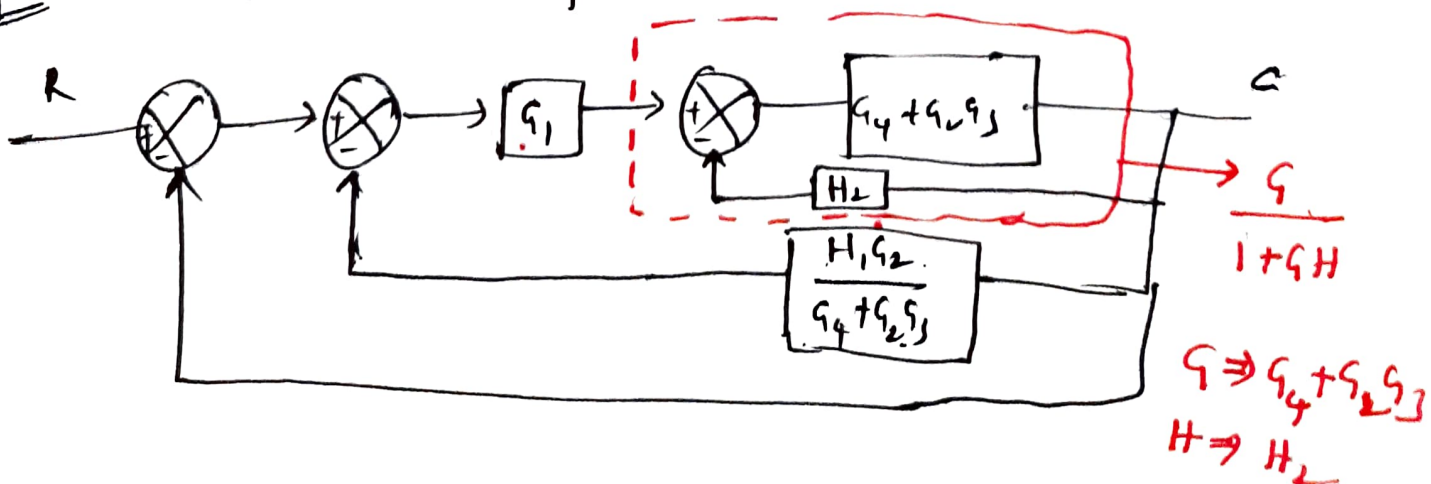
Step 1: Moving the branch point before the block G_2



Step 2: combining blocks in cascade and eliminating parallel blocks.



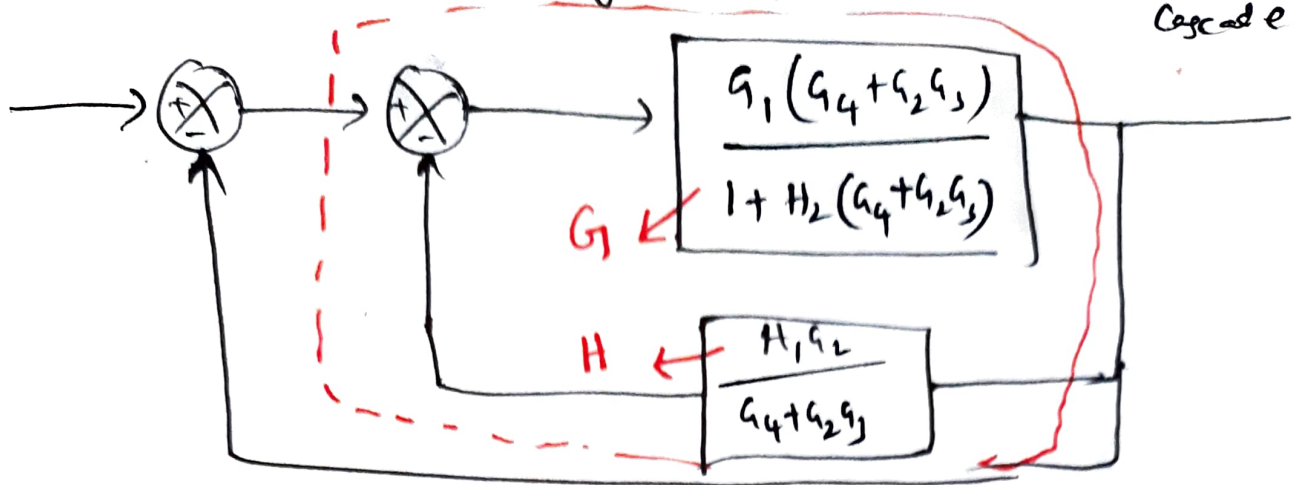
Step 3: moving the branch point after $(G_4 + G_2G_3)$ block



Step 4

Eliminating

negative f/b loop and blocks in cascade



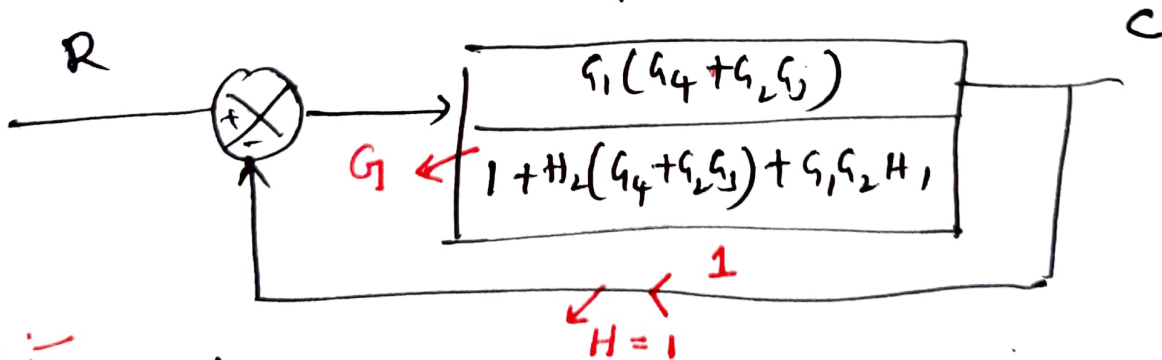
Step 5

Eliminating negative f/b loop

$$\frac{G}{1+GH}$$

$$= \frac{G_1(G_4+G_2G_3)}{1+H_2(G_4+G_2G_3)} \cdot \left[\frac{H_1G_2}{G_4+G_2G_3} \right]$$

$$= \frac{G_1(G_4+G_2G_3)}{1+H_2(G_4+G_2G_3)+G_1G_2H_1}$$



Step 6

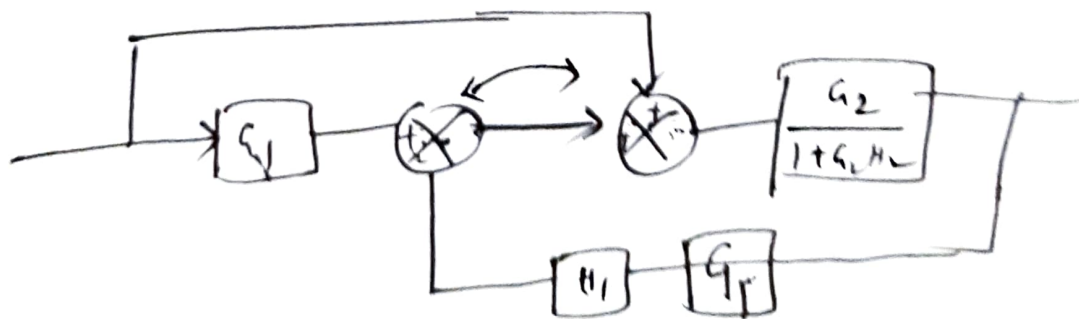
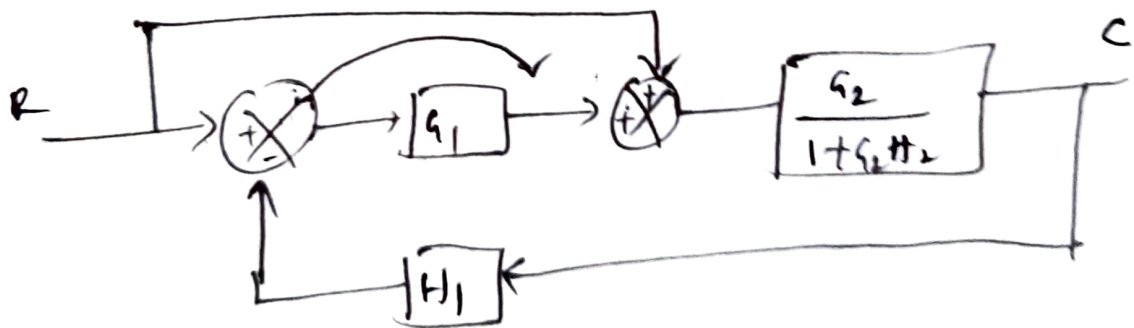
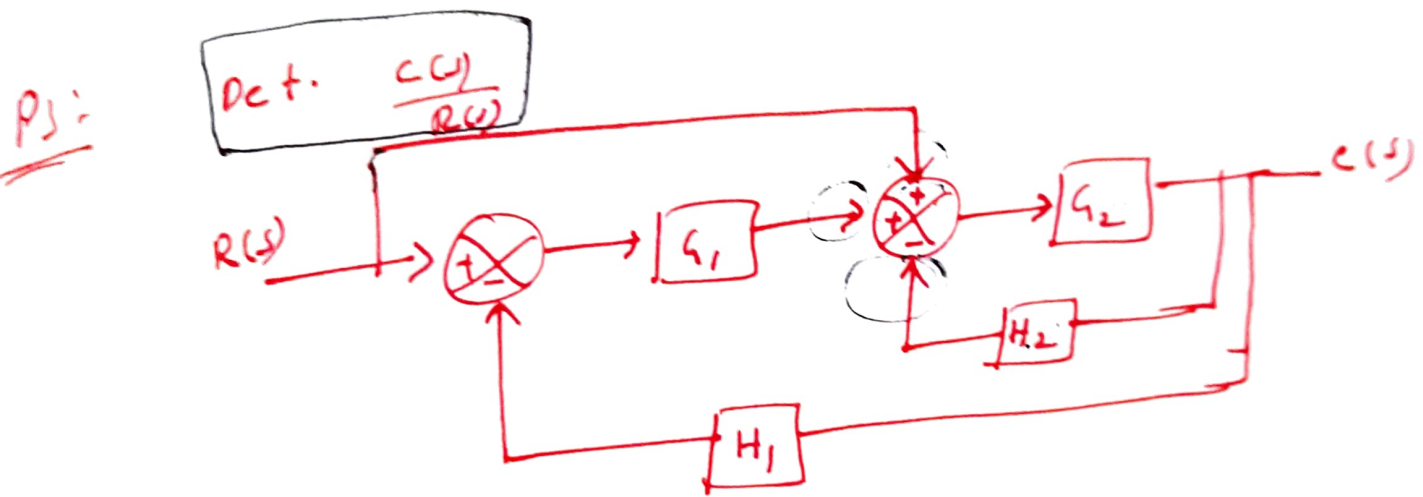
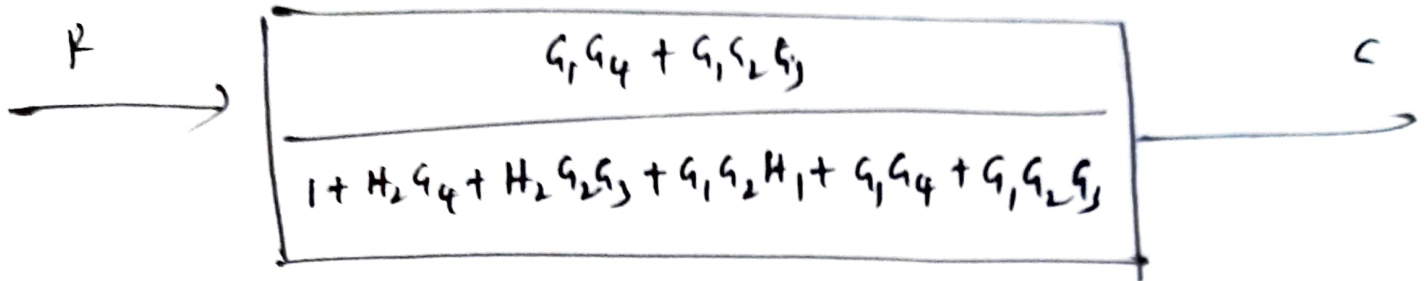
Eliminating

negative f/b loop

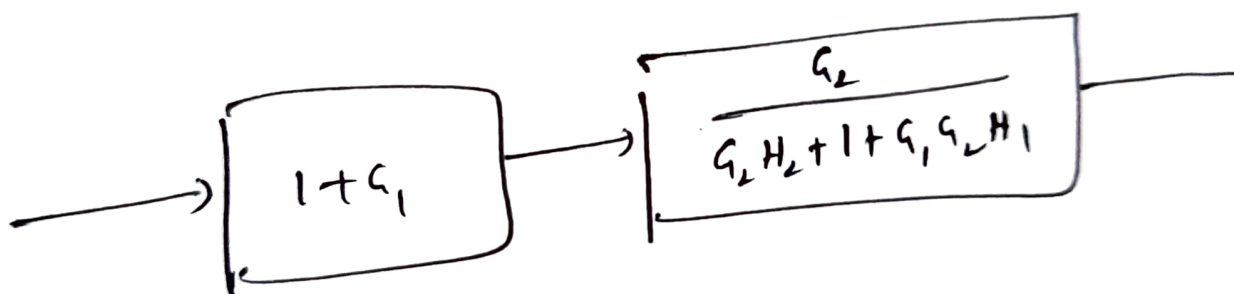
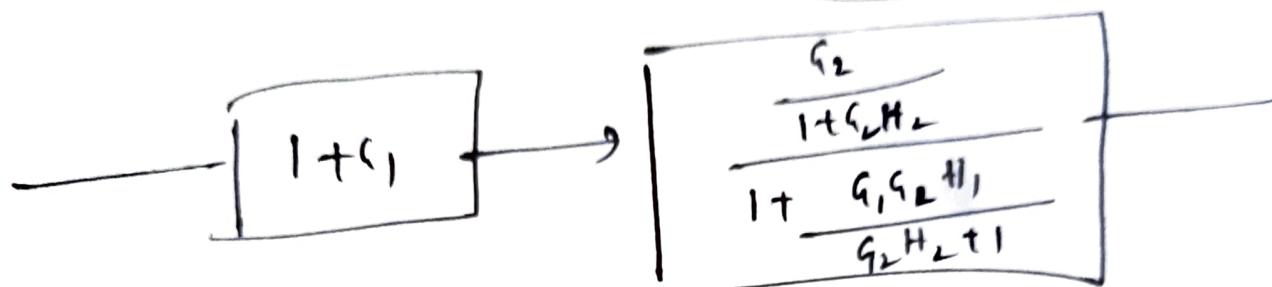
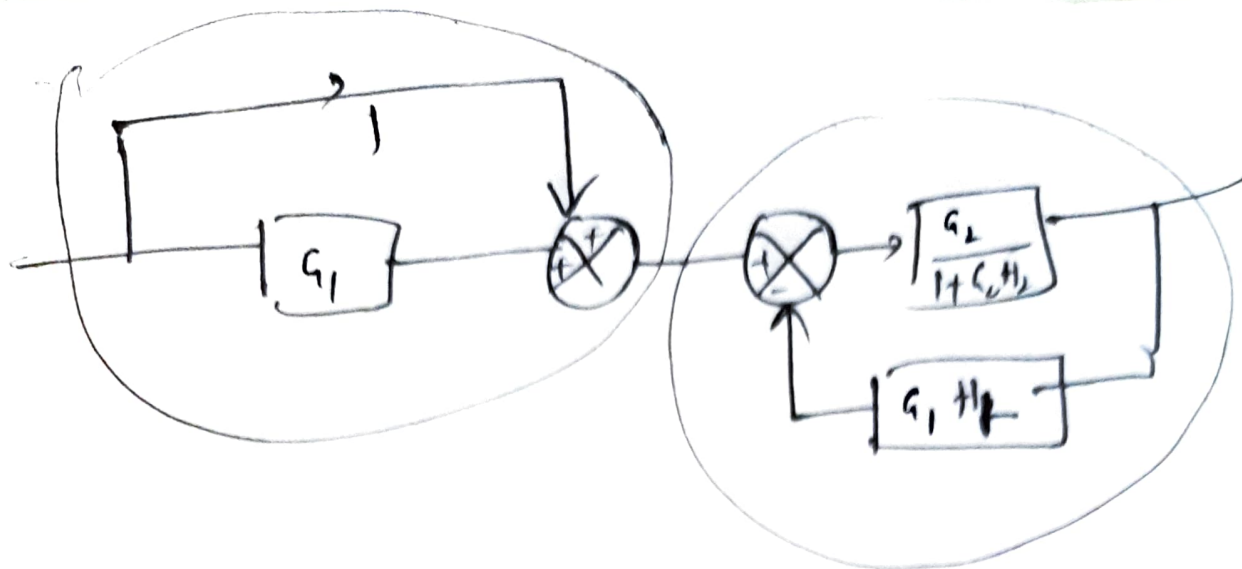
$$\frac{C}{R} = \frac{G}{1+GH}$$

$$= \frac{G_1(G_4+G_2G_3)}{1+H_2(G_4+G_2G_3)+G_1G_2H_1} \cdot \frac{1}{1+\frac{G_1(G_4+G_2G_3)}{1+H_2(G_4+G_2G_3)+G_1G_2H_1}}$$

$$\frac{C}{R} = \frac{G_1(G_4 + G_2G_3)}{1 + H_2(G_4 + G_2G_3) + G_1G_2H_1 + G_1(G_4 + G_2G_3)}$$

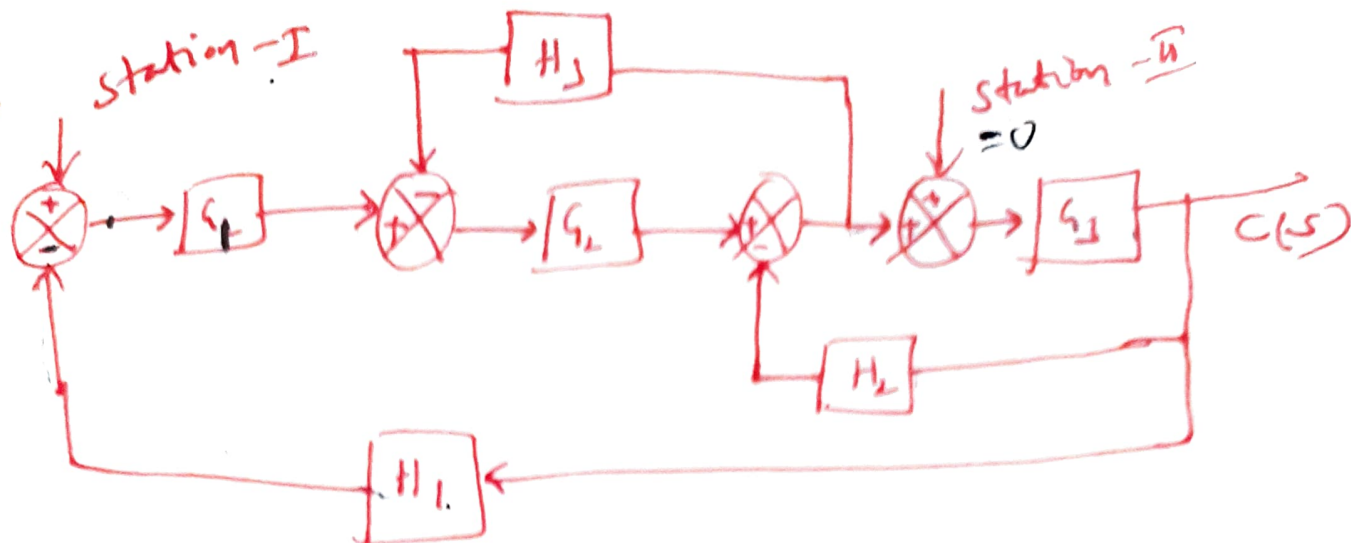


in the



$$\frac{C}{R} = \frac{G_2 (1+G_1)}{1+G_2 H_2 + G_1 G_2 H_2}$$

19/3/21



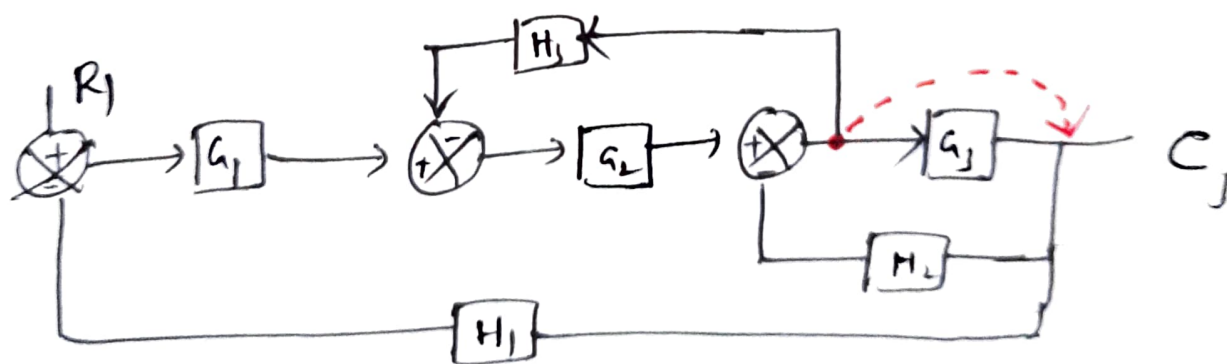
Evaluate closed loop T.F when input R is

- i) at station I ii) station-II

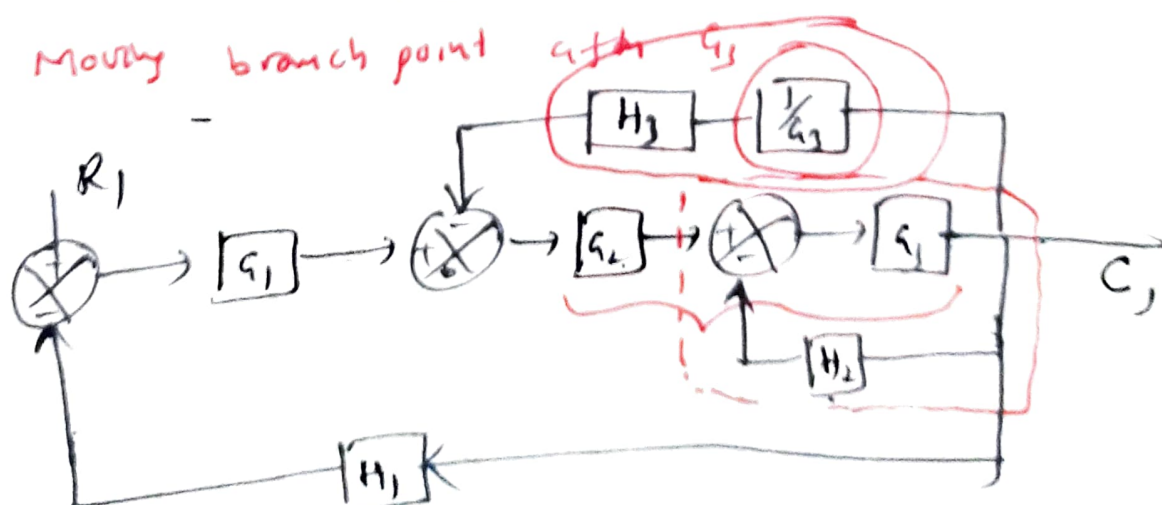
Solution:

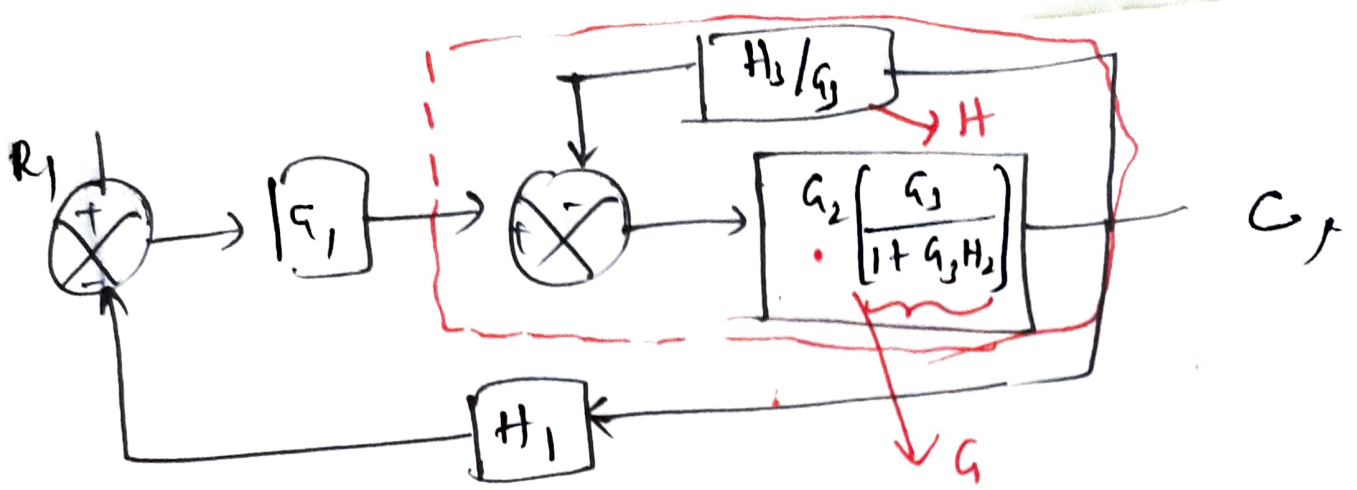
- i) when input R is at station-I

We have to consider i/p at station-I is zero.
Let that o/p is C_1 and input R_1 .

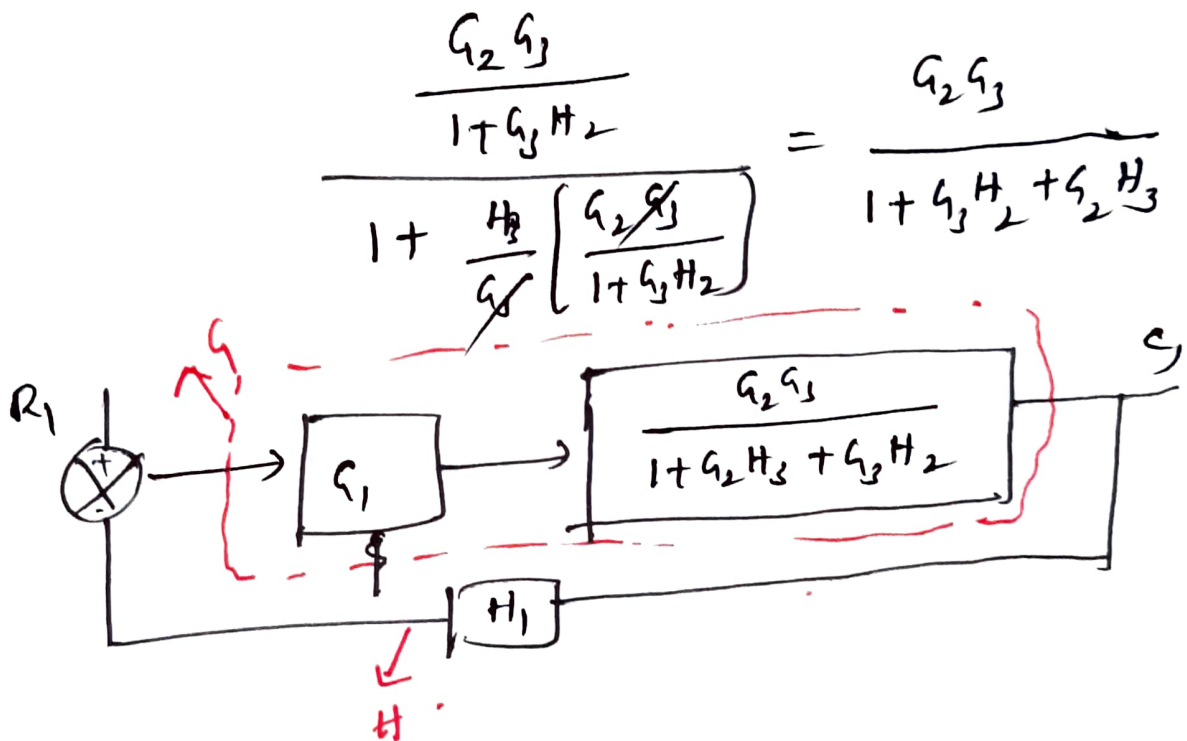


step 1: Moving branch point after G_3



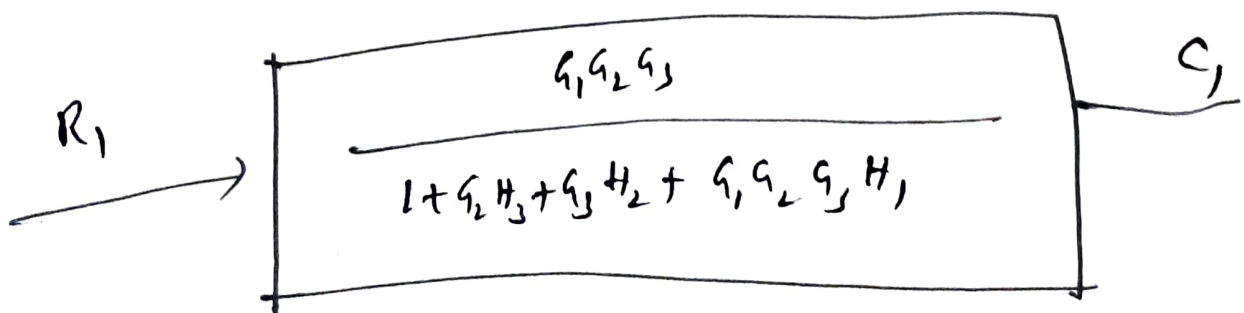


step 3: Eliminate negative feedback.



step 4

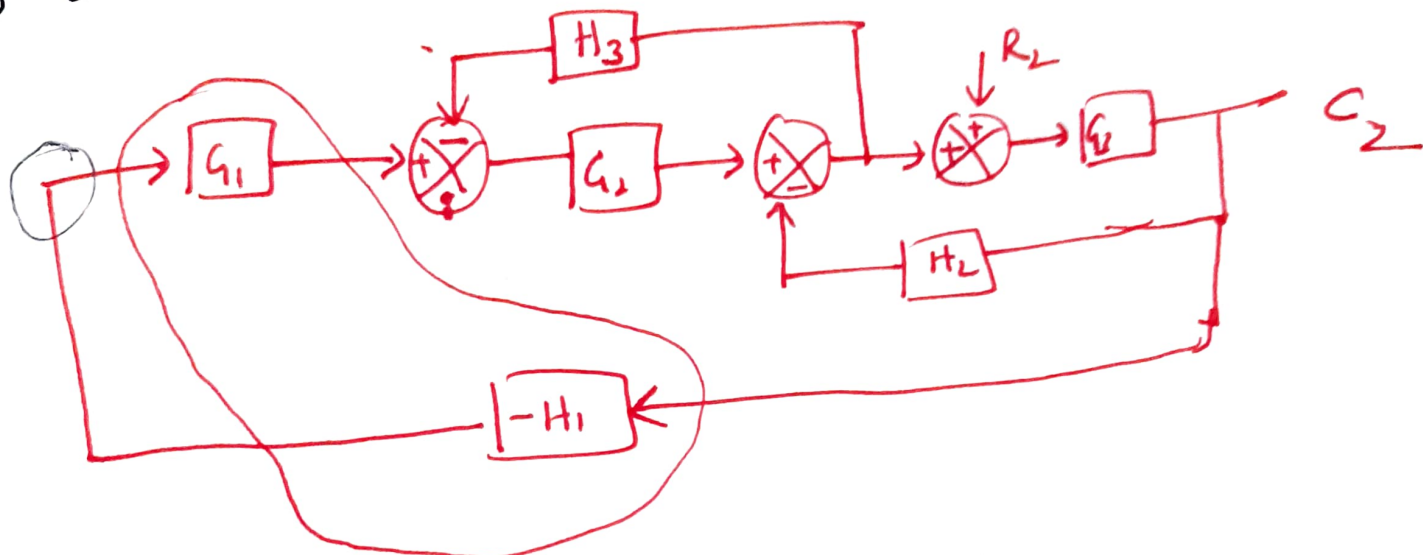
$$\frac{C_1}{R_1} = \frac{\frac{G_1 G_2 G_3}{1 + G_2 H_3 + G_3 H_2}}{1 + \frac{H_1 G_1 G_2 G_3}{1 + G_2 H_3 + G_3 H_2}} = \frac{G_1 G_2 G_3}{1 + G_2 H_3 + G_3 H_2 + G_1 G_2 G_3 H_1}$$



case ii)

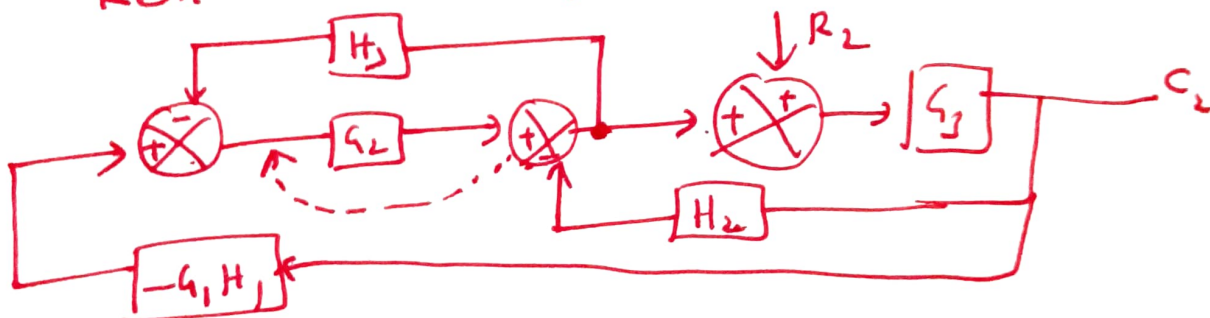
Let input at station-II

when i/p at station-II, consider i/p at station-I is zero. and let o/p is C_2 , and V_p is R_2



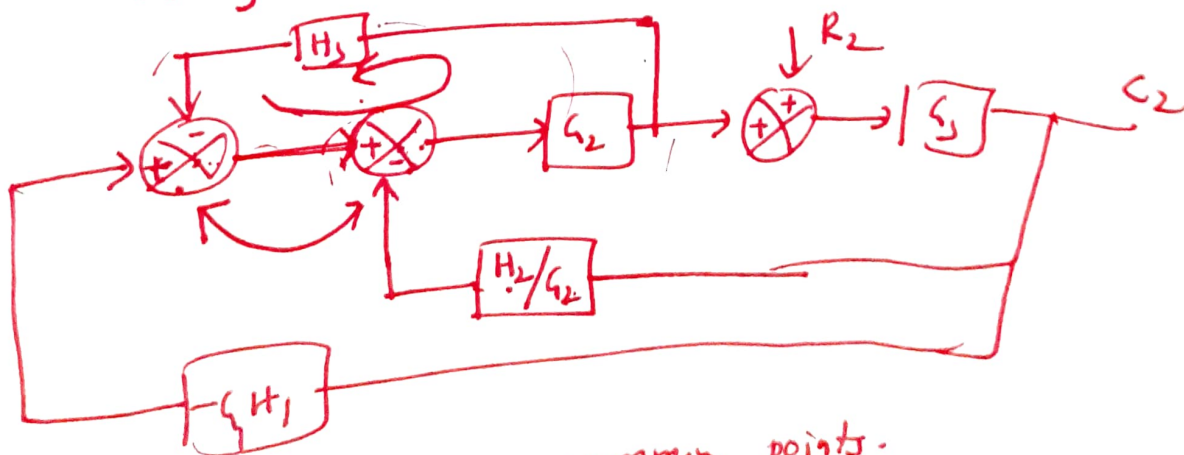
step 1)

Remove the blocks in series



step 2)

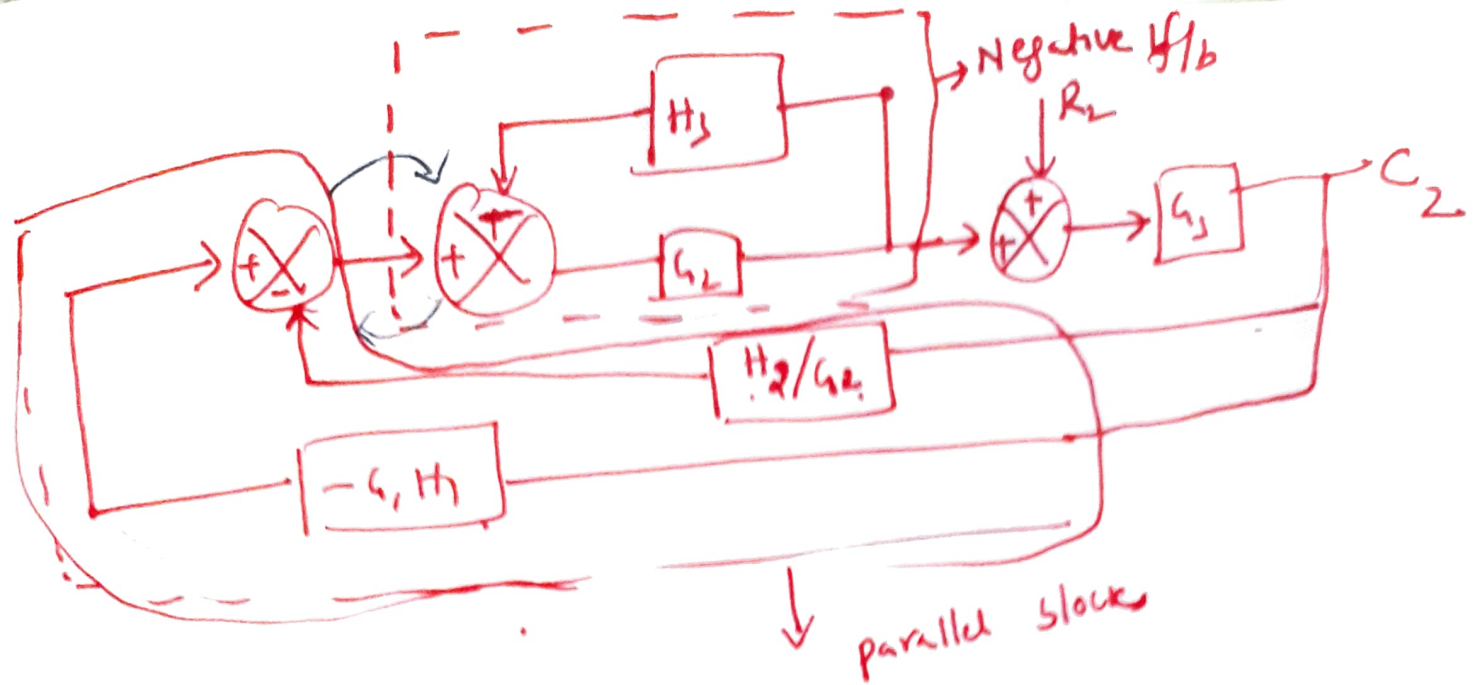
Moving branch summing point before G_2



step 3)

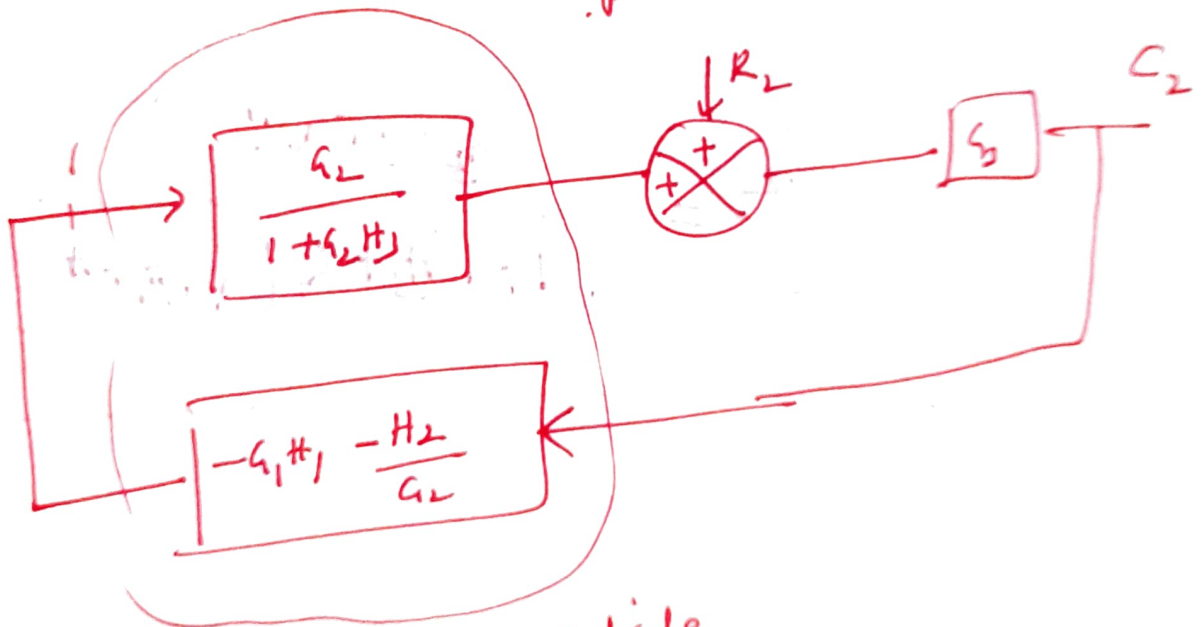
Interchanging of summing points.



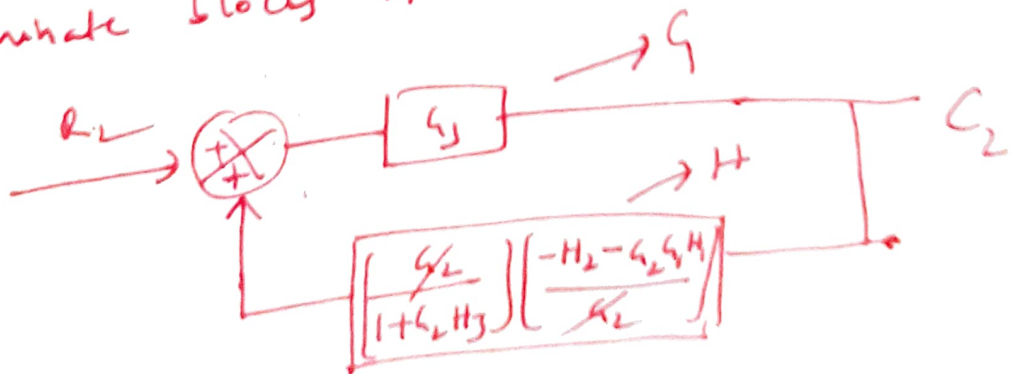


Step

Eliminate negative feedback loop and parallel blocks.



→ eliminate blocks in cascade



→ eliminate positive f/b loop

$$\frac{C_2}{R_2} = \frac{G}{1-GH} = \frac{G_3}{1 + \frac{(H_2 + G_2G_1H_1)}{1+G_2H_1} \cdot G_3}$$

$$\frac{C_2}{R_2} = \frac{G_3 (1 + G_2 H_3)}{1 + G_2 H_3 + G_3 (G_1 G_2 H_1 + H_2)}$$

The signal flow graph is used to represent the control system graphically and it was developed by S.J. Mason.

A signal flow graph is a diagram that represents a set of simultaneous linear algebraic equations. By taking Laplace transform, the time domain differential equations governing a control system can be transferred to a set of algebraic equations in s-domain. The signal flow graph of the system can be constructed using these equations.

It should be noted that the signal flow graph approach and the block diagram approach yield the same information. The advantage in signal flow graph method is that, using Mason's gain formula the overall gain of the system can be computed easily. This method is simpler than the tedious block diagram reduction techniques.

The signal flow graph depicts the flow of signals from one point of a system to another and gives the relationships among the signals. A signal flow graph consists of a network in which nodes are connected by directed branches. Each node represents a system variable and each branch connected between two nodes acts as a signal multiplier. Each branch has a gain or transmittance. When the signal pass through a branch, it gets multiplied by the gain of the branch.

In a signal flow graph, the signal flows in only one direction. The direction of signal flow is indicated by an arrow placed on the branch and the gain (multiplication factor) is indicated along the branch.

EXPLANATION OF TERMS USED IN SIGNAL FLOW GRAPH

- Node** : A node is a point representing a variable or signal.
- Branch** : A branch is directed line segment joining two nodes. The arrow on the branch indicates the direction of signal flow and the gain of a branch is the transmittance.
- Transmittance** : The gain acquired by the signal when it travels from one node to another is called transmittance. The transmittance can be real or complex.
- Input node (Source)** : It is a node that has only outgoing branches.
- Output node (Sink)** : It is a node that has only incoming branches.
- Mixed node** : It is a node that has both incoming and outgoing branches.
- Path** : A path is a traversal of connected branches in the direction of the branch arrows. The path should not cross a node more than once.
- Open path** : A open path starts at a node and ends at another node.
- Closed path** : Closed path starts and ends at same node.
- Forward path** : It is a path from an input node to an output node that does not cross any node more than once.
- Forward path gain** : It is the product of the branch transmittances (gains) of a forward path.
- Individual loop** : It is a closed path starting from a node and after passing through a certain part of a graph arrives at same node without crossing any node more than once.
- Loop gain** : It is the product of the branch transmittances (gains) of a loop.
- Non-touching Loops** : If the loops does not have a common node then they are said to be non-touching loops.

PROPERTIES OF SIGNAL FLOW GRAPH

The basic properties of signal flow graph are the following :

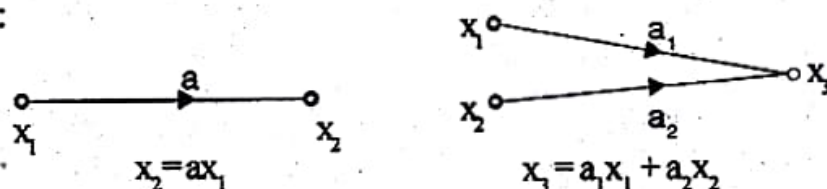
- (i) The algebraic equations which are used to construct signal flow graph must be in the form of cause and effect relationship.
- (ii) Signal flow graph is applicable to linear systems only.
- (iii) A node in the signal flow graph represents the variable or signal.
- (iv) A node adds the signals of all incoming branches and transmits the sum to all outgoing branches.
- (v) A mixed node which has both incoming and outgoing signals can be treated as an output node by adding an outgoing branch of unity transmittance.
- (vi) A branch indicates functional dependence of one signal on the other.
- (vii) The signals travel along branches only in the marked direction and when it travels it gets multiplied by the gain or transmittance of the branch.
- (viii) The signal flow graph of system is not unique. By rearranging the system equations different types of signal flow graphs can be drawn for a given system.

SIGNAL FLOW GRAPH ALGEBRA

Signal flow graph for a system can be reduced to obtain the transfer function of the system using the following rules. The guideline in developing the rules for signal flow graph algebra is that the signal at a node is given by sum of all incoming signals.

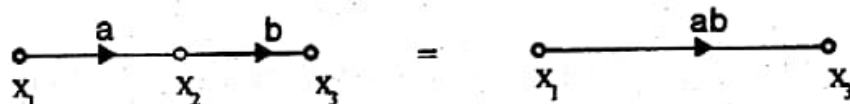
Rule 1 : Incoming signal to a node through a branch is given by the product of a signal at previous node and the gain of the branch.

Example:



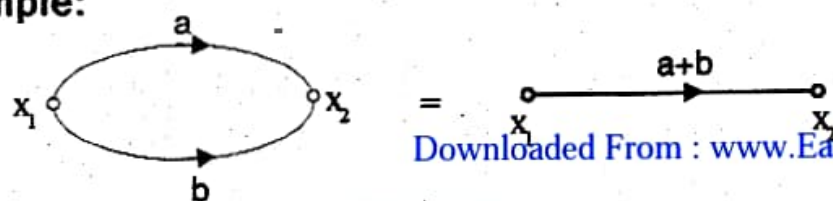
Rule 2 : Cascaded branches can be combined to give a single branch whose transmittance is equal to the product of individual branch transmittance.

Example:



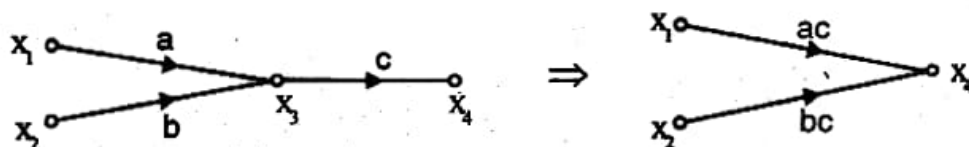
Rule 3 : Parallel branches may be represented by single branch whose transmittance is the sum of individual branch transmittances.

Example:



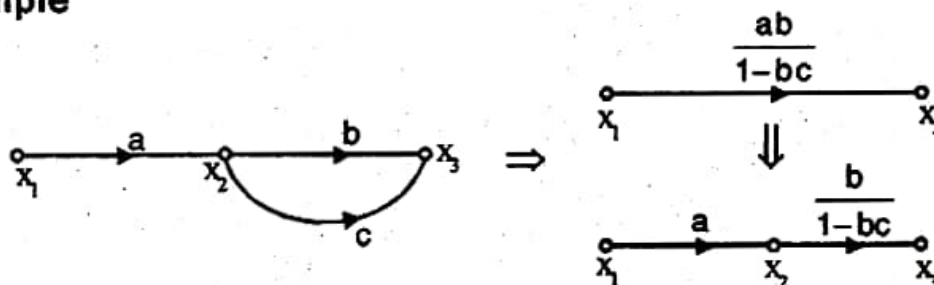
Rule 4 : A mixed node can be eliminated by multiplying the transmittance of outgoing branch (from the mixed node) to the transmittance of all incoming branches to the mixed node.

Example



Rule 5 : A loop may be eliminated by writing equations at the input and output node and rearranging the equations to find the ratio of output to input. This ratio gives the gain of resultant branch.

Example



Proof:

$$x_2 = ax_1 + cx_3 ; \quad x_3 = bx_2$$

Put, $x_2 = ax_1 + cx_3$ in the equation for x_3 .

$$\therefore x_3 = b(ax_1 + cx_3) \Rightarrow x_3 = abx_1 + bcx_3 \Rightarrow x_3 - bcx_3 = abx_1 \Rightarrow x_3(1 - bc) = abx_1$$

$$\therefore \frac{x_3}{x_1} = \frac{ab}{1 - bc}$$

SIGNAL FLOW GRAPH REDUCTION

The signal flow graph of a system can be reduced either by using the rules of a signal flow graph algebra or by using Mason's gain formula.

For signal flow graph reduction using the rules of signal flow graph, write equations at every node and then rearrange these equations to get the ratio of output and input (transfer function).

The signal flow graph reduction by above method will be time consuming and tedious. S.J. Mason has developed a simple procedure to determine the transfer function of the system represented as a signal flow graph. He has developed a formula called by his name **Mason's gain formula** which can be directly used to find the transfer function of the system.

MASON'S GAIN FORMULA

The Mason's gain formula is used to determine the transfer function of the system from the signal flow graph of the system.

Let, $R(s)$ = Input to the system

$C(s)$ = Output of the system

Now, Transfer function of the system, $T(s) = \frac{C(s)}{R(s)}$ (1.34)

Mason's gain formula states the overall gain of the system [transfer function] as follows,

$$\text{Overall gain, } T = \frac{1}{\Delta} \sum_k P_k \Delta_k \quad \text{Downloaded From : } \text{www.EasyEngineering.net} \quad \text{.....(1.35)}$$

- where, $T = T(s) =$ Transfer function of the system
 $P_K =$ Forward path gain of K^{th} forward path
 $K =$ Number of forward paths in the signal flow graph
 $\Delta = 1 - (\text{Sum of individual loop gains})$
 $+ \left(\text{Sum of gain products of all possible combinations of two non-touching loops} \right)$
 $- \left(\text{Sum of gain products of all possible combinations of three non-touching loops} \right)$
 $+ \dots\dots\dots$
 $\Delta_K = \Delta$ for that part of the graph which is not touching K^{th} forward path

CONSTRUCTING SIGNAL FLOW GRAPH FOR CONTROL SYSTEMS

A control system can be represented diagrammatically by signal flow graph. The differential equations governing the system are used to construct the signal flow graph. The following procedure can be used to construct the signal flow graph of a system.

1. Take Laplace transform of the differential equations governing the system in order to convert them to algebraic equations in s-domain.
2. The constants and variables of the s-domain equations are identified.
3. From the working knowledge of the system, the variables are identified as input, output and intermediate variables.
4. For each variable a node is assigned in signal flow graph and constants are assigned as the gain or transmittance of the branches connecting the nodes.
5. For each equation a signal flow graph is drawn and then they are interconnected to give overall signal flow graph of the system.

PROCEDURE FOR CONVERTING BLOCK DIAGRAM TO SIGNAL FLOW GRAPH

The signal flow graph and block diagram of a system provides the same information but there is no standard procedure for reducing the block diagram to find the transfer function of the system. Also the block diagram reduction technique will be tedious and it is difficult to choose the rule to be applied for simplification. Hence it will be easier if the block diagram is converted to signal flow graph and **Mason's gain formula** is applied to find the transfer function. The following procedure can be used to convert block diagram to signal flow graph.

1. Assume nodes at input, output, at every summing point, at every branch point and in between cascaded blocks.
2. Draw the nodes separately as small circles and number the circles in the order 1, 2, 3, 4, etc.
3. From the block diagram find the gain between each node in the main forward path and connect all the corresponding circles by straight line and mark the gain between the nodes.
4. Draw the feed forward paths between various nodes and mark the gain of feed forward path along with sign.
5. Draw the feedback paths between various nodes and mark the gain of feedback with sign.

Signal flow graph Reduction :-

Mason's gain formula

It is used to find transfer function of the system from the given signal flow graph.

Let $R(s) = \text{Input}$

$C(s) = \text{output}$

$$\text{T.F., } T(s) = \frac{C(s)}{R(s)}$$

By using Mason's gain formula, we have

$$\text{overall gain } T = \frac{1}{\Delta} \sum_k P_k \Delta_k$$

where $T = T(s)$ T.F of the system

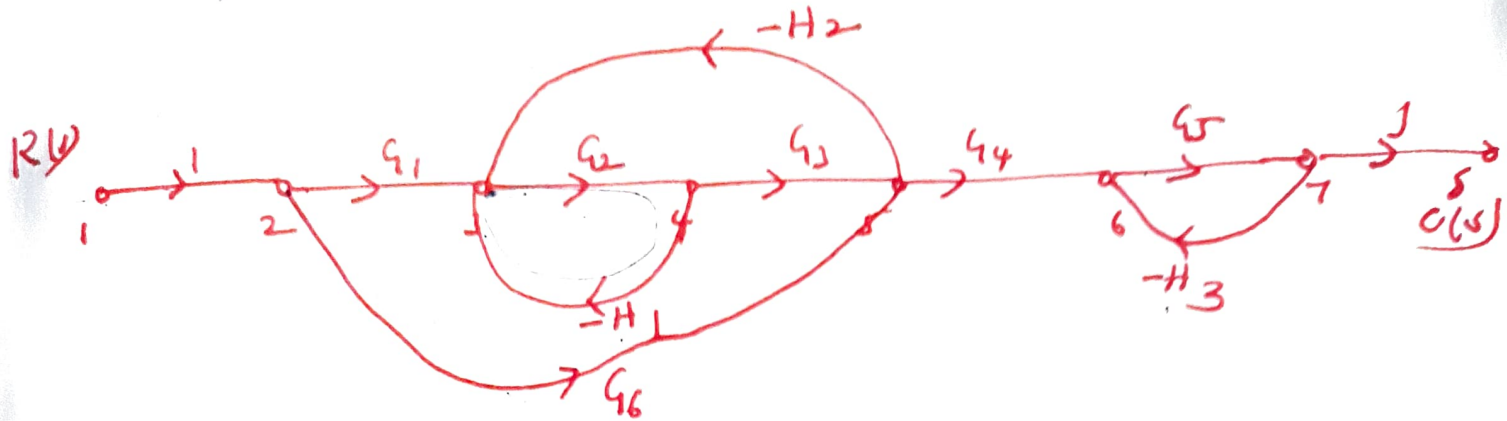
$K = \text{No. of forward paths.}$

$P_k = \text{forward path gain of } k^{\text{th}} \text{ forward path.}$

$$\Delta = 1 - \left[\text{sum of individual loop gains} \right] + \left[\text{sum of gain products of all possible combinations of two non-touching loops} \right] - \left[\text{sum of gain products of three non-touching loops} \right] + \left[\text{sum of gain products of four non-touching loops} \right] - \dots$$

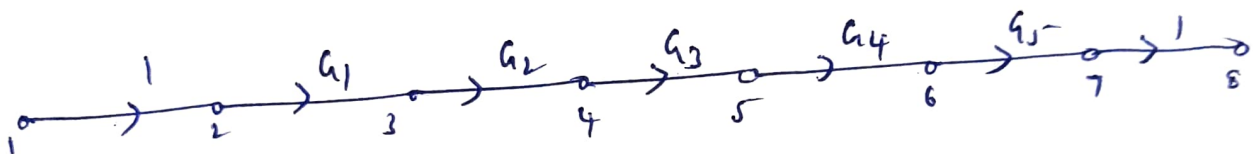
$\Delta_k = \Delta$, for that part of graph which is not touching to k^{th} forward path.

P1: Find the overall gain of the system whose SFG is shown in fig.



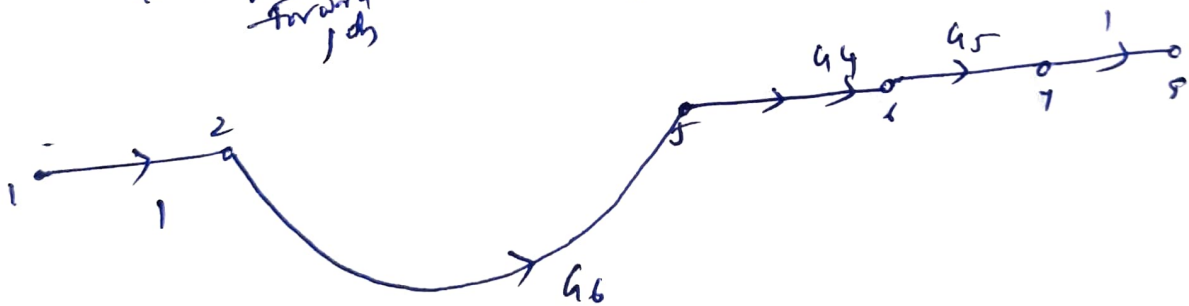
Solution

Forward paths:



Gain of 1st forward path

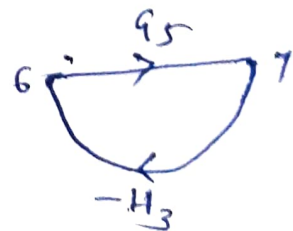
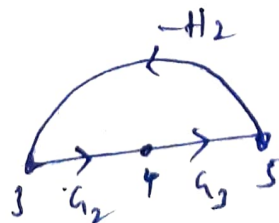
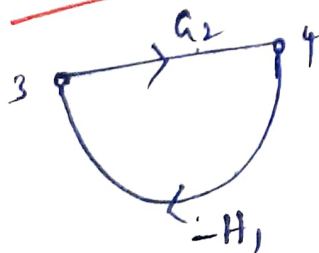
$$P_1 = G_1 G_2 G_3 G_4 G_5$$



$$P_2 = G_6 \cdot G_4 \cdot G_5$$

No. of forward paths (K) = 2

Indivisible loops:



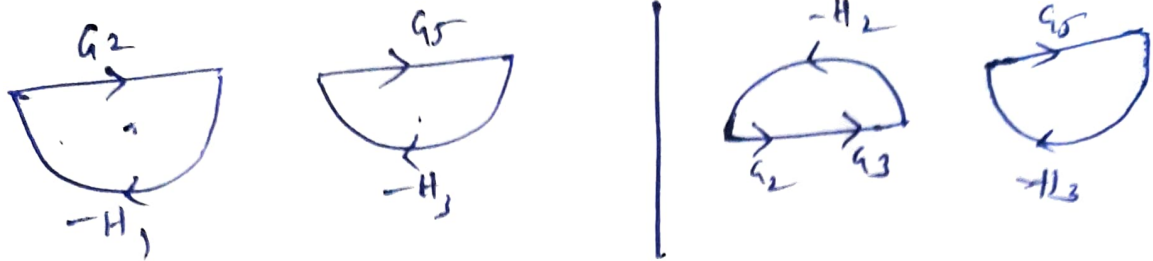
$$L_1 = -G_2 \cdot H_1$$

$$L_2 = -G_2 \cdot G_3 \cdot H_2$$

$$L_3 = -G_5 \cdot H_3$$

Non touching loops

There are two pairs of two non touching loops



$$L_{13} = G_2 G_5 H_1 H_3$$

$$L_{23} = G_2 G_3 G_5 H_2 H_3$$

Calculation of Δ and Δ_K :

$$\Delta = 1 - \left[\text{sum of individual loop gains} \right] + \left[\text{product of sum of two non touching} \right]$$

$$= 1 - [L_1 + L_2 + L_3] + [L_{13} + L_{23}]$$

$$= 1 + G_2 H_1 + G_2 G_3 H_2 + G_5 H_3 + G_2 G_5 H_1 H_3 + G_2 G_3 G_5 H_2 H_3$$

$$\Delta_1 = 1 - 0 = 1$$

$$\Delta_2 = 1 + L_1 = 1 + G_2 H_1$$

T.F.:

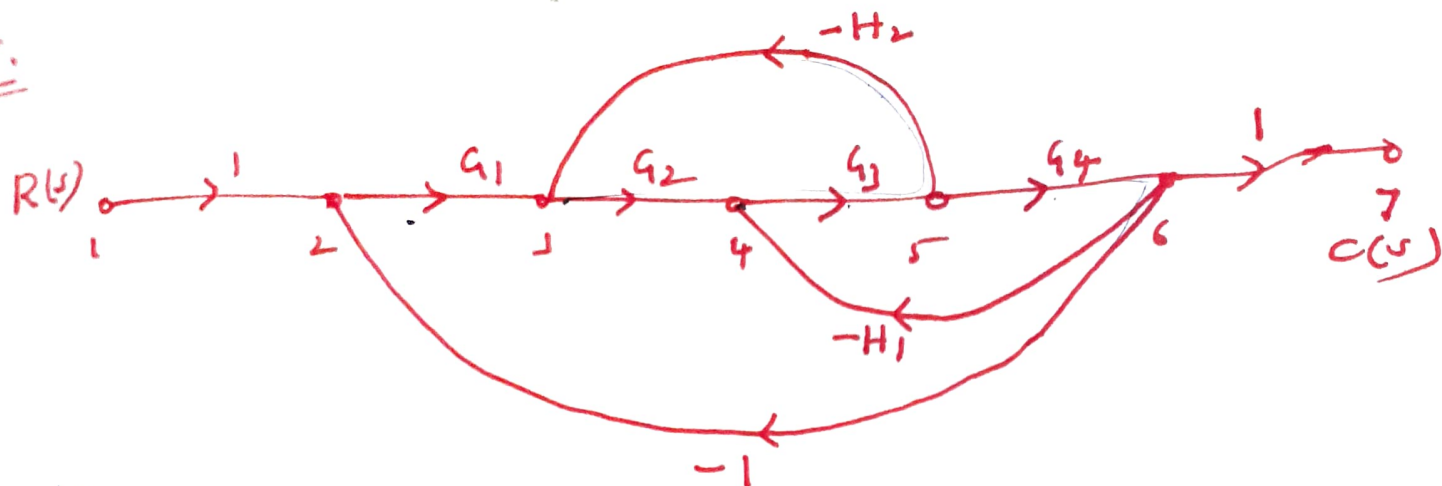
$$T = \frac{1}{\Delta} \sum_K P_K \Delta_K$$

$$K = 2$$

$$= \frac{1}{\Delta} [P_1 \Delta_1 + P_2 \Delta_2]$$

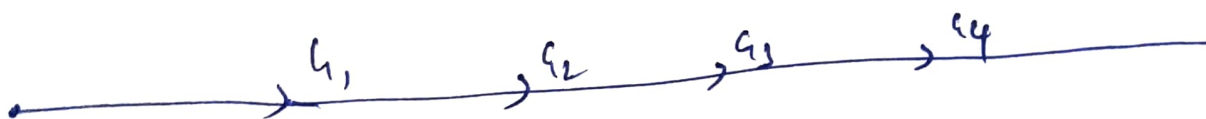
$$T = \frac{G_1 G_2 G_3 G_4 G_5 + G_4 G_5 G_6 (1 + G_2 H_1)}{1 + G_2 H_1 + G_2 G_3 H_2 + G_5 H_3 + G_2 G_5 H_1 H_3 + G_2 G_3 G_5 H_2 H_3}$$

P2:



Sol:

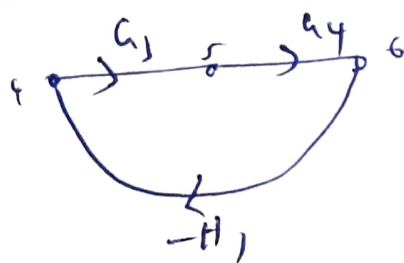
Forward path



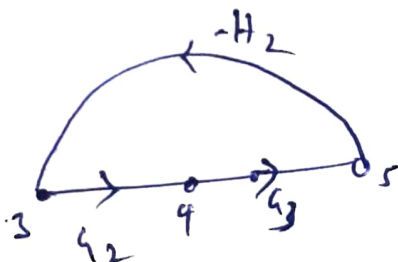
$$P_1 = G_1 G_2 G_3 G_4$$

$$K=1$$

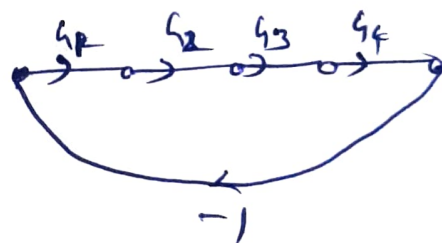
Individual loops



$$L_1 = -G_3 G_4 H_1$$



$$L_2 = -G_2 G_3 H_2$$



$$L_3 = -G_1 G_2 G_3 G_4$$

Non touching loops:

Zero.

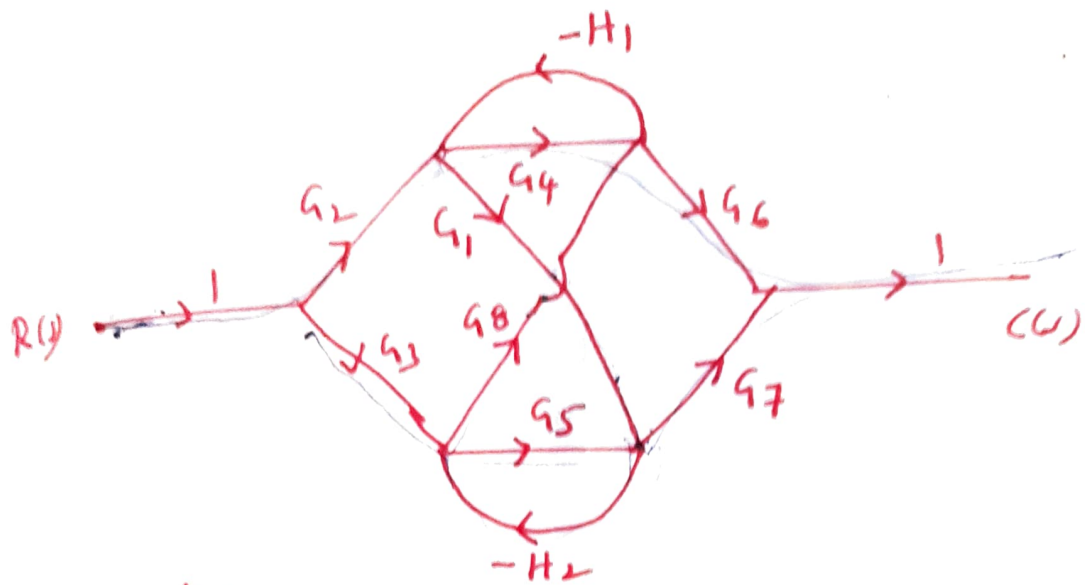
There is No Non touching loop pairs.

$$\Delta = 1 - [L_1 + L_2 + L_3] + 0$$

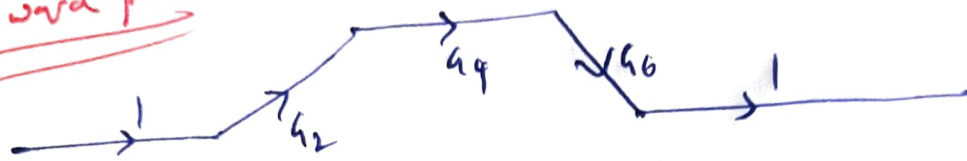
$$= 1 + G_3 G_4 H_1 + G_2 G_3 H_2 + G_1 G_2 G_3 G_4$$

$$\Delta_1 = 1 - 0 = 1$$

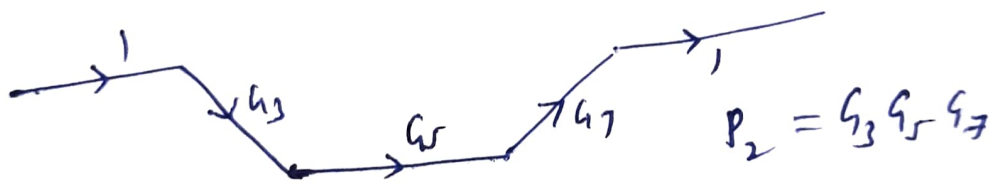
$$T.F = \frac{P_1 \Delta_1}{\Delta} = \frac{G_1 G_2 G_3 G_4}{1 + G_3 G_4 H_1 + G_2 G_3 H_2 + G_1 G_2 G_3 G_4}$$



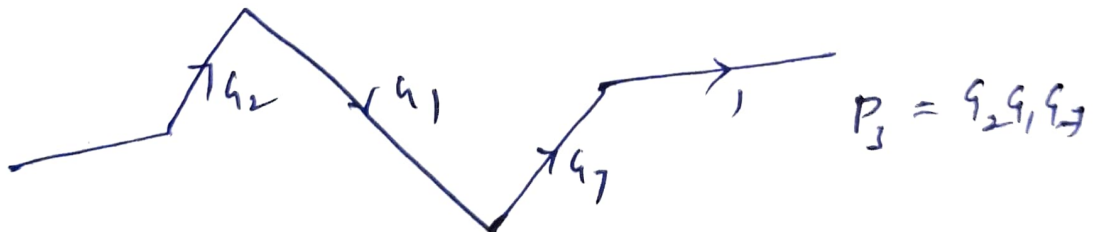
sol: Forward paths.



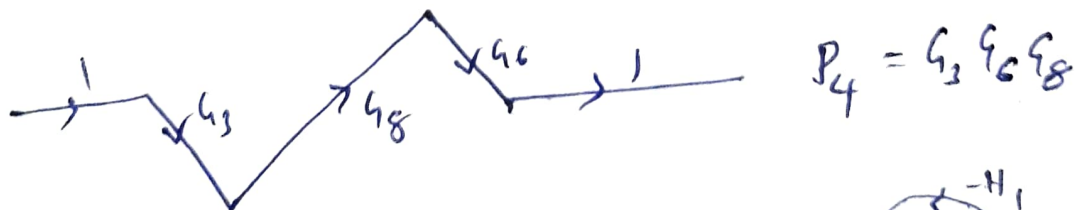
$$P_1 = G_2 G_4 G_6$$



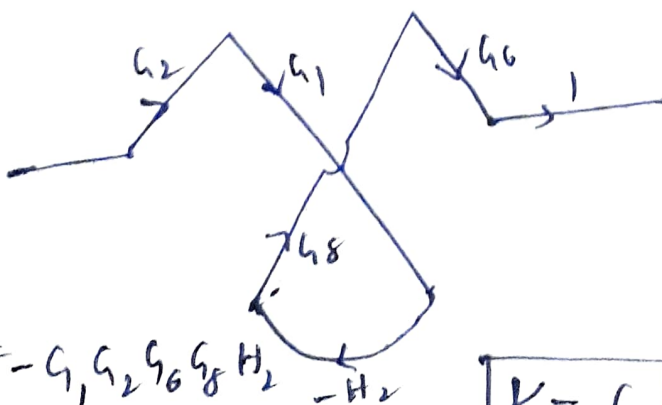
$$P_2 = G_3 G_5 G_7$$



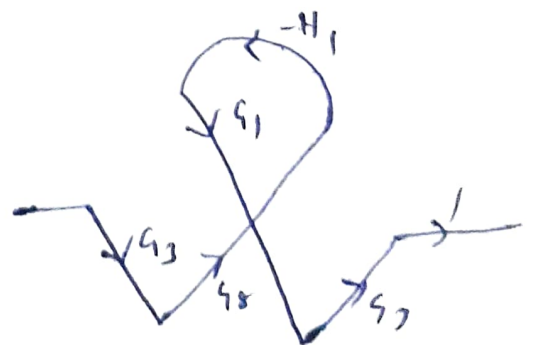
$$P_3 = G_2 G_1 G_7$$



$$P_4 = G_3 G_8 G_6$$



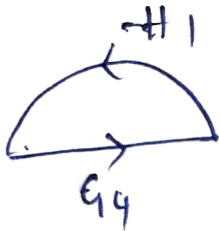
$$P_5 = -G_1 G_2 G_6 G_8 H_2$$



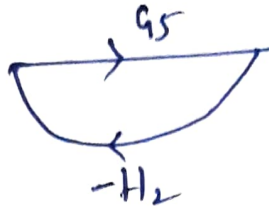
$$P_6 = -G_1 G_3 G_7 G_8 H_1$$

$$K = 6$$

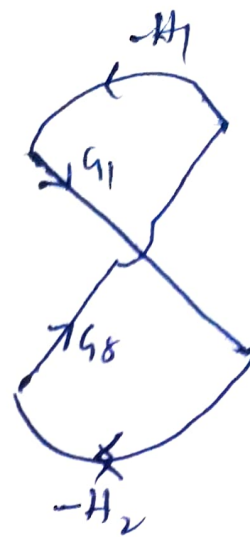
Individual loops:



$$L_1 = -G_4 H_1$$



$$L_2 = -G_5 H_2$$



$$L_3 = G_1 G_8 H_1 H_2$$

Non touching loop pairs:

$$L_1 L_2 = G_4 G_5 H_1 H_2$$

Δ & Δ_K

$$\Delta = 1 - [L_1 + L_2 + L_3] + L_1 L_2$$

$$= 1 + G_4 H_1 + G_5 H_2 - G_1 G_8 H_1 H_2 + G_4 G_5 H_1 H_2$$

$$\Delta_1 = 1 + G_5 H_2$$

$$\Delta_4 = 1 - 0 = 1$$

$$\Delta_2 = 1 + G_4 H_1$$

$$\Delta_5 = 1 - 0 = 1$$

$$\Delta_3 = 1 - 0 = 1$$

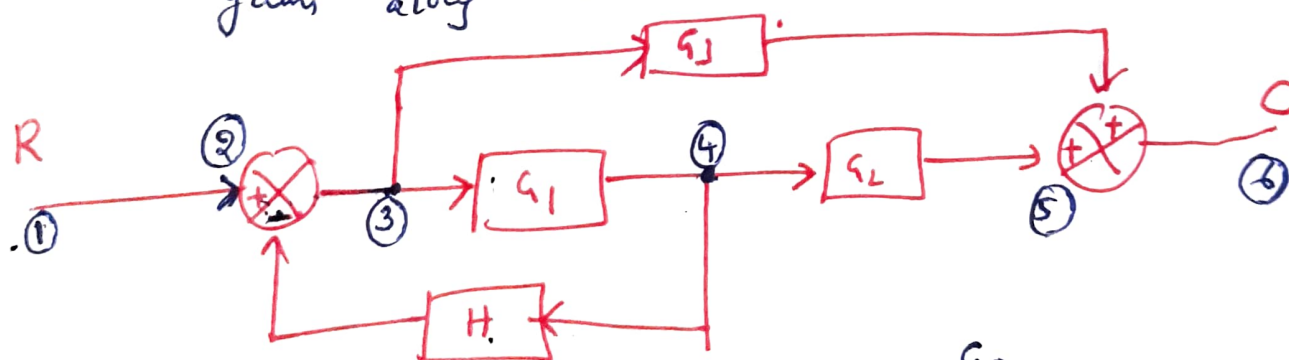
$$\Delta_6 = 1 - 0 = 1$$

$$T.F = \frac{P_1 \Delta_1 + P_2 \Delta_2 + P_3 \Delta_3 + P_4 \Delta_4 + P_5 \Delta_5 + P_6 \Delta_6}{\Delta}$$

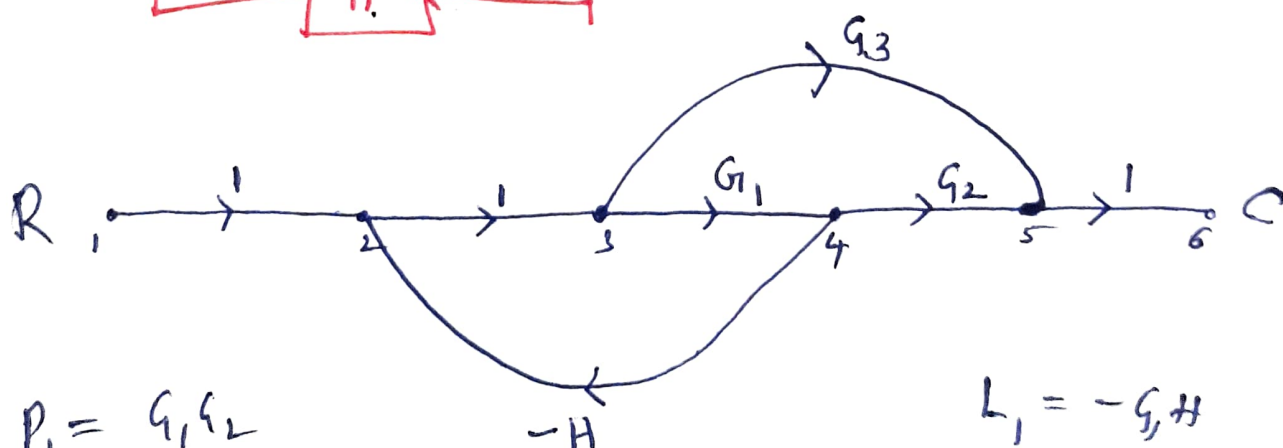
$$T.F = \frac{G_2 G_4 G_6 (1 + G_5 H_2) + G_3 G_5 G_7 (1 + G_4 H_1) + G_1 G_2 G_7 + G_3 G_6 G_8 - G_1 G_3 G_7 G_8 H_1 - G_1 G_2 G_6 G_8 H_2}{1 + G_4 H_1 + G_5 H_2 - G_1 G_8 H_1 H_2 + G_4 G_5 H_1 H_2}$$

procedure for converting block diagrams to signal flow graph.

- 1) Assume nodes at i/p, o/p, at every summing point and at every branch point.
- 2) Draw the nodes separately as small circles and indicate numbers like 1, 2, 3, ...
- 3) From the block diagram find the gain b/w branches and indicate in signal flow graph.
- 4) Draw the forward paths and loops by indicating gain along the branches.



Sol:



$$P_1 = G_1 G_2$$

$$P_2 = G_3$$

$$K=2$$

$$\Delta = 1 + G_1 H$$

$$\Delta_1 = \Delta_2 = 1$$

$$L_1 = -G_1 H$$

$$T.F = \frac{P_1 \Delta_1 + P_2 \Delta_2}{\Delta} = \frac{G_1 G_2 + G_3}{1 + G_1 H}$$

unit - II

Time Response Analysis

o/p of closed loop system as a function of

time. $C(t)$

$$\frac{C(s)}{R(s)} \checkmark$$

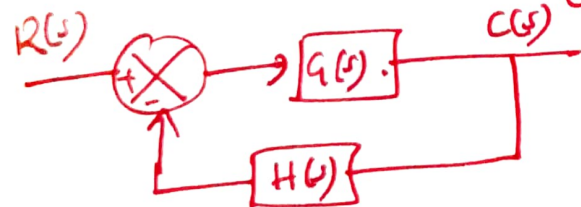
↓ Transfer function

closed loop T.F $\frac{C(s)}{R(s)} = \frac{G(s)}{1+G(s)H(s)} = M(s)$

$$C(s) = M(s) \cdot R(s)$$

$$C(t) = \mathcal{L}^{-1}\{C(s)\} = \mathcal{L}^{-1}\{M(s) \cdot R(s)\}$$

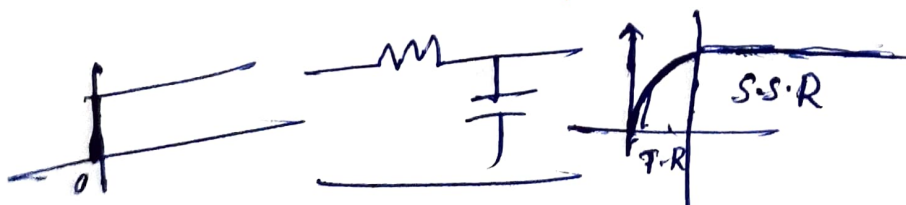
where $M(s) = \frac{G(s)}{1+G(s)H(s)}$



Time Response $\begin{cases} \rightarrow \text{transient Response} \\ \rightarrow \text{steady state Response} \end{cases}$

Transient Response: It is response of the system when the i/p changes from one state to other.

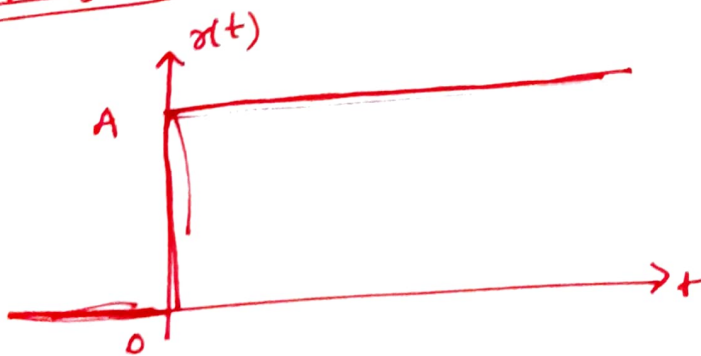
Steady state Response: It is the response as time 't' approaches to ∞ .



TEST Signals:-

- 1) step signal.
- 2) Ramp signal
- 3) parabolic signal
- 4) sinusoidal signal
- 5) Impulse signal.

Step Signals



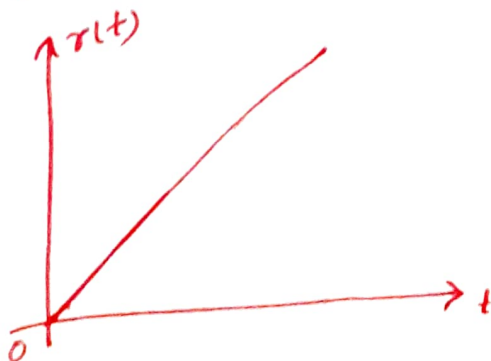
$$x(t) = A, t \geq 0 \\ = 0, t < 0$$

$$R(s) = \frac{A}{s}$$

If it is unit step signal
 $x(t) = 1, t \geq 0$
 $= 0, t < 0$

$$R(s) = \frac{1}{s}$$

Ramp signals



$$x(t) = At, t \geq 0 \\ = 0, t < 0$$

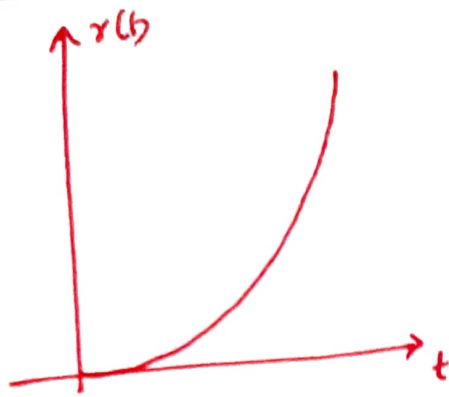
$$R(s) = \frac{A}{s^2}$$

For unit Ramp

$$x(t) = t$$

$$R(s) = \frac{1}{s^2}$$

parabolic signal:

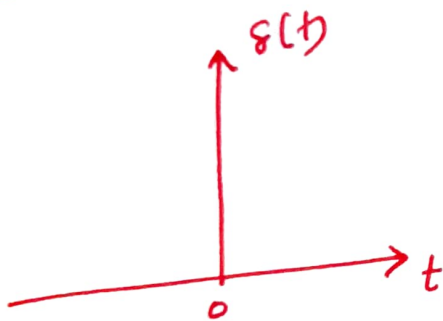


$$r(t) = At^2/2, \quad t \geq 0$$

$$= 0, \quad t < 0$$

$$\mathcal{L}\{r(t)\} = R(s) = 1/s^3$$

Impulse signal



$$\delta(t) = \infty, \quad t = 0$$

$$= 0, \quad t \neq 0$$

$$\int_{-\infty}^{\infty} \delta(t) dt = 1$$

$$\mathcal{L}\{\delta(t)\} = 1$$

impulse response The response of the system, when the input as impulse signal is called impulse response

$$\text{Impulse Response } m(t) = \mathcal{L}^{-1}\{R(s)M(s)\}$$

$$= \mathcal{L}^{-1}\{1 \cdot M(s)\}$$

$$m(t) = \mathcal{L}^{-1}\{M(s)\}$$

where $M(s) = \frac{G(s)}{1 + G(s)H(s)}$

"order" of the system:

$$\text{T.F.} \quad T(s) = \frac{P(s)}{Q(s)} = \frac{b_0 s^m + b_1 s^{m-1} + \dots + b_m}{a_0 s^n + a_1 s^{n-1} + \dots + a_n}$$

$P(s)$ — numerator

$Q(s)$ — denominator

The order = Max. power of s in $Q(s)$
Total no. of poles

$$Q(s) \rightarrow a_0 s^n + a_1 s^{n-1} + \dots + a_n$$

$$n=0 \Rightarrow \text{order} = 0$$

$$n=1 \Rightarrow \text{order} = 1$$

$$\vdots$$
$$n=10 \Rightarrow \text{order} = 10$$

Eg:- $\frac{1}{s+1} \Rightarrow \text{order} = 1$

$$\frac{1}{s^2+2s+2} \Rightarrow \text{order} = 2$$

poles: Roots of denominator polynomials

zeros: Roots of numerator polynomials

Eg: $T(s) = \frac{s+1}{(s+2)(s+3)} \Rightarrow \text{zeros: } s=-1$
 $\text{poles: } s=-2, -3$

$$T(s) = \frac{s}{(s^2+2s+1)}$$

$$\text{zeros: } s=0$$

$$\text{poles: } s^2+2s+1=0$$
$$(s+1)^2=0 \quad s=-1, -1$$

"Type" Number of System:-

Number of poles at origin will give type number. It is defined for only open loop transfer function

Eg:
$$\frac{s+1}{s^3(s+2)(s+4)}$$

Zero $s = -1$
Poles $s = 0, 0, 0, -2, -4$

order : 5.
Type : 3

$$\frac{1}{(s+4)(s+3)} \Rightarrow \text{order : 2}$$

type : 0

Review of partial fractions Expansion:

Case i) Function with separate poles

ii) Function with multiple poles

iii) " " complex poles.

Case i)

$$T(s) = \frac{K}{s(s+p_1)(s+p_2)} = \frac{A}{s} + \frac{B}{s+p_1} + \frac{C}{s+p_2}$$

$$A = T(s) \times s \Big|_{s=0}$$

$$B = T(s) \times (s+p_1) \Big|_{s=-p_1}$$

$$C = T(s) \times (s+p_2) \Big|_{s=-p_2}$$

Case (i)

$$T(s) = \frac{K}{s(s+p_1)(s+p_2)^2}$$

$$= \frac{A}{s} + \frac{B}{s+p_1} + \frac{C}{s+p_2} + \frac{D}{(s+p_2)^2}$$

A, B, C

D $\rightarrow$ compare s^2 coeff -

Case (ii)

$$\text{Let } T(s) = \frac{K}{(s+p_1)(s^2+bs+c)}$$

$$= \frac{A}{s+p_1} + \frac{Bs+C}{s^2+bs+c}$$

Eg:

$$T(s) = \frac{1}{(s+1)(s+2)}$$

$$\frac{1}{(s+1)(s+2)} = \frac{A}{s+1} + \frac{B}{s+2}$$

$$1 = A(s+2) + B(s+1) \rightarrow (1)$$

$$\text{Let } \underline{s=-1}$$

$$1 = A + 0$$
$$\boxed{A=1}$$

$$\underline{s=-2}$$

$$1 = -B$$

$$\boxed{B=-1}$$

$$\boxed{T(s) = \frac{1}{s+1} - \frac{1}{s+2}}$$

$$T(s) = \frac{1}{(s+1)(s+2)^2}$$

$$\frac{1}{(s+1)(s+2)^2} = \frac{A}{s+1} + \frac{B}{s+2} + \frac{C}{(s+2)^2}$$

$$1 = A(s+2)^2 + B(s+2)(s+1) + C(s+1)$$

$$\underline{s=-1}$$

$$\boxed{A=1}$$

$$\underline{s=-2}$$

$$C=-1$$

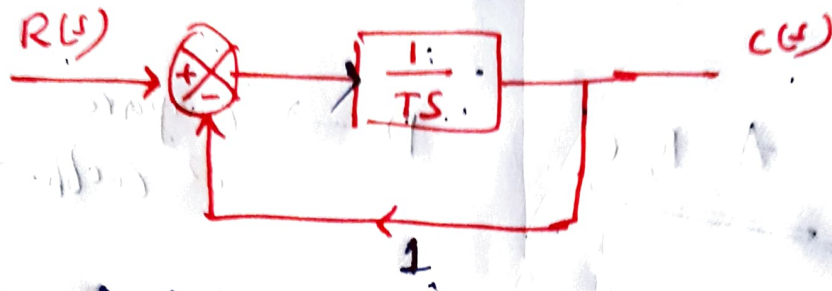
compare s^2 coeff

$$A+B=0 \Rightarrow \boxed{B=-1}$$

23/03/2021

Time Response of first order systems for unit step

Input:



consider first order closed loop system with unity feedback

$$\frac{C(s)}{R(s)} = \frac{G}{1+GH} \quad \checkmark$$

$$\text{Here } G = \frac{1}{Ts}, \quad H = 1$$

$$\frac{C(s)}{R(s)} = \frac{\frac{1}{Ts}}{1 + \frac{1}{Ts}} = \frac{1}{Ts + 1}$$

CLTF closed loop Transfer function $\left[\frac{C(s)}{R(s)} = \frac{1}{Ts + 1} \right] \rightarrow \textcircled{1}$

Let input $r(t)$ = unit step

$$r(t) = \begin{cases} 1, & t \geq 0 \\ 0, & t < 0 \end{cases}$$

Then $R(s) = \frac{1}{s}$

From ①

$$C(s) = \frac{1}{Ts+1} \cdot R(s)$$

$$= \frac{1}{(Ts+1)} \cdot \frac{1}{s}$$

$$[\because R(s) = \frac{1}{s}]$$

$$= \frac{1}{s(Ts+1)}$$

$$C(s) = \frac{1}{Ts(s+1/T)} = \frac{1/T}{s(s+1/T)}$$

$$C(s) = \frac{1/T}{s(s+1/T)} = \frac{A}{s} + \frac{B}{s+1/T}$$

$$\frac{1/T}{s(s+1/T)} = \frac{A(s+1/T) + Bs}{s(s+1/T)}$$

$$1/T = A(s+1/T) + Bs$$

when $s = -1/T$

$$1/T = 0 + B(-1/T)$$

$$\boxed{B = -1}$$

when $s = 0$

$$1/T = A(1/T) + 0$$

$$\Rightarrow \boxed{A = 1}$$

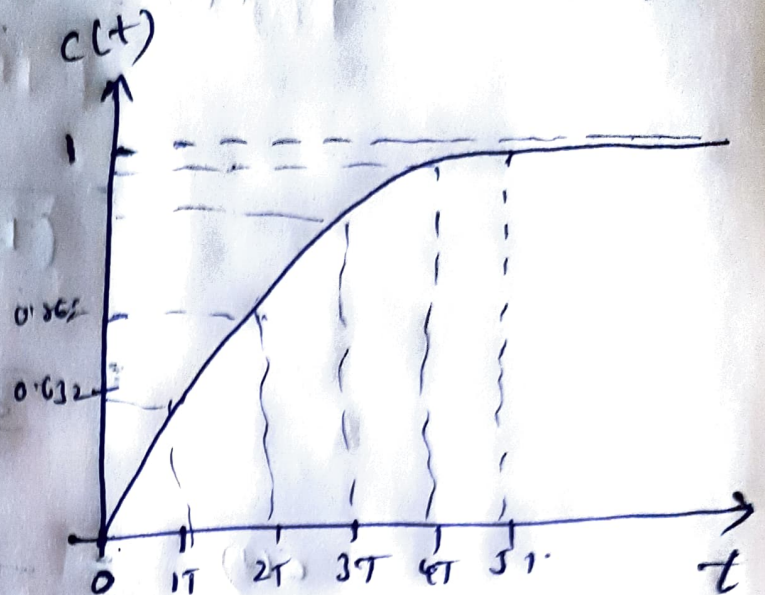
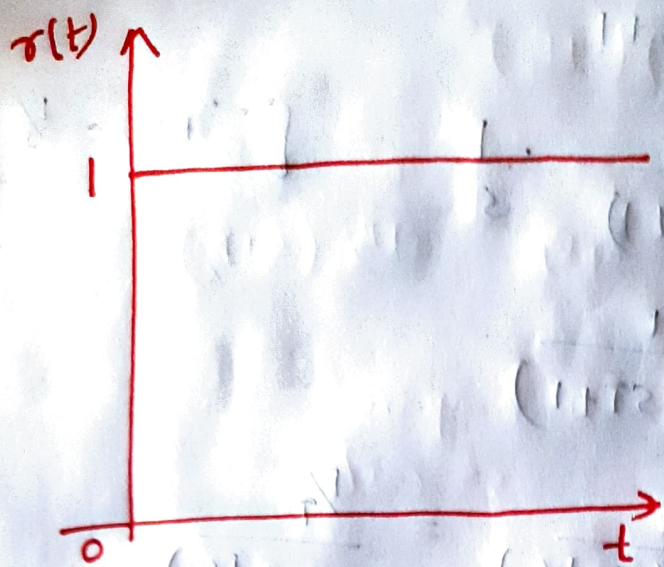
$$\mathcal{L}^{-1}\left\{\frac{1}{s}\right\} = 1$$

$$\mathcal{L}^{-1}\left\{\frac{1}{s+a}\right\} = e^{-at}$$

$$C(s) = \frac{1}{s} - \frac{1}{s+1/T}$$

$$c(t) = \mathcal{L}^{-1}\{C(s)\} = \left[1 - e^{-t/T}\right], t \geq 0$$

$$\boxed{c(t) = 1 - e^{-t/T}}$$



$$c(t) = 1 - e^{-t/T}$$

When $t=0$, $c(t) = 1 - e^0 = 1 - 1 = 0$

$t=T$, $c(t) = 1 - e^{-1} = 0.632$

$t=2T$, $c(t) = 1 - e^{-2} = 0.865$

$t=3T$, $c(t) = 1 - e^{-3} = 0.95$

$t=4T$, $c(t) = 1 - e^{-4} = 0.9817$

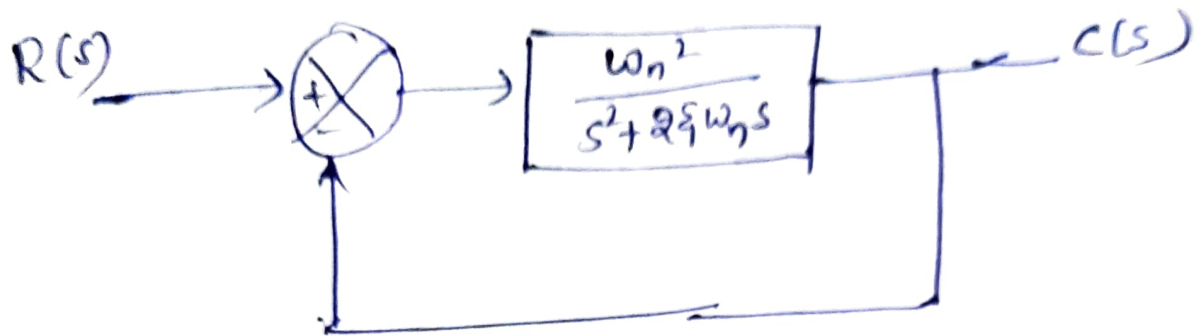
$t=5T$, $c(t) = 1 - e^{-5} = 0.993$

$t=\infty$, $c(t) = 1 - e^{-\infty} = 1$

' T ' is called Time Constant of the system

Second order systems:-

The standard closed loop second order system is shown in fig.



$$\frac{C(s)}{R(s)} = \frac{\frac{\omega_n^2}{s^2 + 2\xi\omega_n s}}{1 + \frac{\omega_n^2}{s^2 + 2\xi\omega_n s}}$$

standard
second
order
CLTF

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\xi\omega_n s + \omega_n^2}$$

$\omega_n \rightarrow$ undamped natural frequency
rad/sec

$\xi \rightarrow$ Damping Ratio $\boxed{\text{Xeta } \xi}$

Damping ratio is defined as ratio of the actual damping ~~and~~ to the critical damping.

The time response of second order system $C(t)$ will depend on damping ratio.

Depending upon ξ , the system can be classified into 4 cases.

Case (i) : Undamped system ($\xi = 0$)

Case (ii) : Underdamped system ($0 < \xi < 1$)

Case (iii) : Critically damped system ($\xi = 1$)

Case (iv) : Over damped system ($\xi > 1$)

The characteristic equation of the second order system is

$$s^2 + 2\xi\omega_n s + \omega_n^2 = 0$$

Roots are: $s = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ $ax^2 + bx + c = 0$

$$s_1, s_2 = \frac{-2\xi\omega_n \pm \sqrt{4\xi^2\omega_n^2 - 4\omega_n^2}}{2}$$

$$= \frac{-2\xi\omega_n \pm \omega_n \sqrt{\xi^2 - 1}}{2}$$

$$s_1, s_2 = -\xi\omega_n \pm \omega_n \sqrt{\xi^2 - 1}$$

When $\xi = 0$, $s_1, s_2 = \pm j\omega_n$, Roots are purely imaginary
System is undamped

$\xi = 1$, $s_1, s_2 = -\omega_n, -\omega_n$, Roots are real and equal
System is critically damped

$\xi > 1$, $s_1, s_2 = -\xi\omega_n \pm \omega_n \sqrt{\xi^2 - 1}$, Roots are real and unequal
overdamped

$0 < \xi < 1$, $s_1, s_2 = -\xi\omega_n \pm j\omega_n \sqrt{1 - \xi^2}$, complex roots.
underdamped

$$= -\xi\omega_n \pm j\omega_d$$

$$\boxed{\omega_d = \omega_n \sqrt{1 - \xi^2}}$$

Response of Undamped second order system

for unit step input:

The standard second order system CLTF is given by

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

For undamped system, $\zeta = 0$

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + \omega_n^2}$$

when the input is unit step

$$r(t) = u(t)$$

$$R(s) = 1/s$$

Then $C(s) = \frac{\omega_n^2}{s^2 + \omega_n^2} \cdot R(s)$

$$C(s) = \frac{\omega_n^2}{s(s^2 + \omega_n^2)} = \frac{A}{s} + \frac{Bs + C}{s^2 + \omega_n^2}$$

$$\Rightarrow \omega_n^2 = A(s^2 + \omega_n^2) + (Bs + C)s$$

when $s=0 \Rightarrow \boxed{A=1}$

~~$$s = +j\omega_n \Rightarrow \frac{\omega_n^2}{s(s^2 + \omega_n^2)} = \frac{A}{s} + \frac{Bs + C}{s^2 + \omega_n^2}$$~~

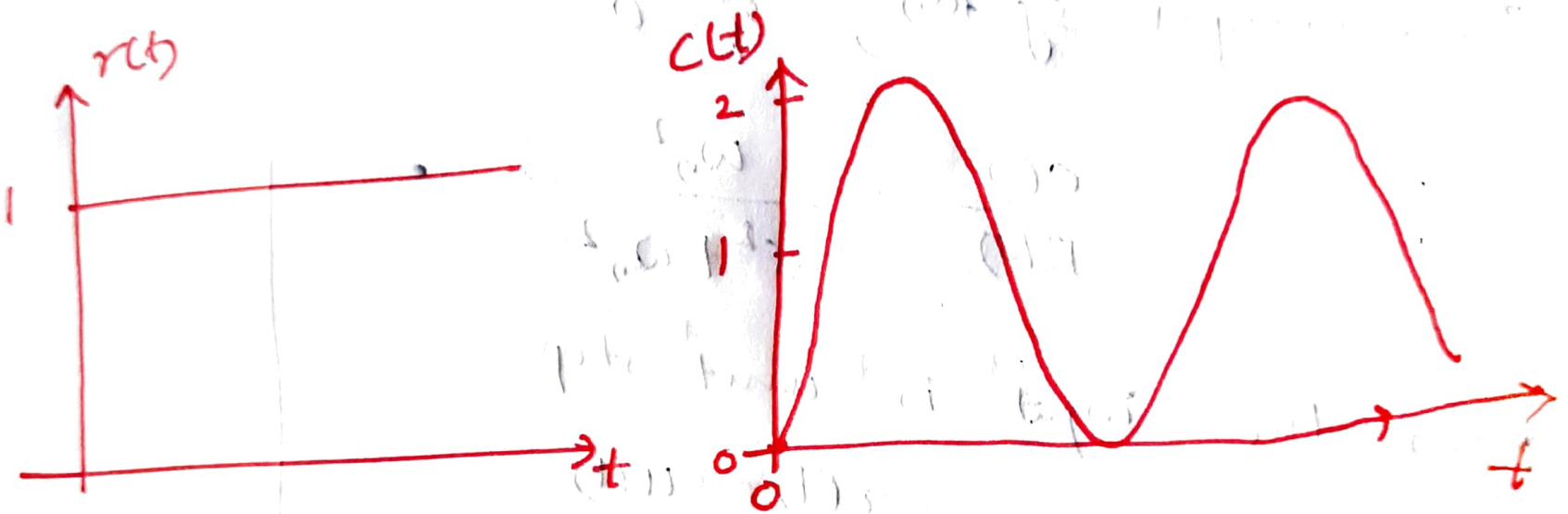
compare s^2 coeff. $A + B = 0 \Rightarrow B = -1$

" s coeff. $\boxed{C=0}$

$$C(s) = \frac{1}{s} - \frac{s}{s^2 + \omega_n^2}$$

Time Response $C(t) = \mathcal{L}^{-1}\{C(s)\}$

$$C(t) = 1 - \cos \omega_n t$$



24/03/21

Response of underdamped system for unit step input:-

consider standard second order system

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\xi\omega_n s + \omega_n^2}$$

For underdamped system, $0 < \xi < 1$ and roots of the denominator are complex conjugate.

$$s_{1/2} = -\xi\omega_n \pm j\omega_n\sqrt{1-\xi^2}$$

$$s = -\xi\omega_n \pm \omega_n\sqrt{1-\xi^2}$$

$$= -\xi\omega_n \pm \omega_n\sqrt{(-1)(1-\xi^2)}$$

Since $\xi < 1$

ξ^2 also less than 1

$1-\xi^2$ is always positive

$$\therefore s = -\xi\omega_n \pm j\omega_n\sqrt{1-\xi^2}$$

$$\therefore s = -\xi\omega_n \pm j\omega_n\sqrt{1-\xi^2}$$

The damped frequency of oscillation $\omega_d = \omega_n\sqrt{1-\xi^2}$

$$s = -\xi\omega_n \pm j\omega_d$$

we have

$$C(s) = \frac{\omega_n^2}{s^2 + 2\xi\omega_n s + \omega_n^2} \cdot R(s)$$

But input is unit step i.e.

$$r(t) = u(t)$$

$$R(s) = \frac{1}{s}$$

$$C(s) = \frac{\omega_n^2}{s^2 + 2\xi\omega_n s + \omega_n^2} \quad \underline{1 \text{ over}}$$

$$C(s) = \frac{\omega_n^2}{s(s^2 + 2\xi\omega_n s + \omega_n^2)} = \frac{A}{s} + \frac{Bs + C}{s^2 + 2\xi\omega_n s + \omega_n^2}$$

$$\omega_n^2 = A(s^2 + 2\xi\omega_n s + \omega_n^2) + (Bs + C)s$$

when s=0

$$\omega_n^2 = A(\omega_n^2) + 0$$

$$\Rightarrow \boxed{A=1}$$

compare 's' coefficients

$$A + B = 0 \Rightarrow \boxed{A = -B}$$

$$\boxed{B = -A = -1}$$

compare 's' coefficients

$$2\xi\omega_n A + C = 0$$

$$C = -A(2\xi\omega_n)$$

$$\boxed{C = -2\xi\omega_n}$$

$$C(s) = \frac{1}{s} + \frac{(-1)s - 2\xi\omega_n}{s^2 + 2\xi\omega_n s + \omega_n^2}$$

$$= \frac{1}{s} - \frac{s + 2\xi\omega_n}{s^2 + 2\xi\omega_n s + \omega_n^2}$$

$$a^2 + 2ab + b^2$$

$$= \frac{1}{s} - \frac{s + 2\xi\omega_n}{s^2 + 2\xi\omega_n s + (\xi\omega_n)^2 - (\xi\omega_n)^2 + \omega_n^2}$$

$$= \frac{1}{s} - \frac{(s + 2\xi\omega_n + \xi\omega_n)}{(s + \xi\omega_n)^2 + \omega_n^2(1 - \xi^2)}$$

$$\mathcal{L}^{-1} \left\{ \frac{s+a}{(s+a)^2 + b^2} \right\} = e^{-at} \cos bt$$

$$\mathcal{L}^{-1} \left\{ \frac{b}{(s+a)^2 + b^2} \right\} = e^{-at} \sin bt$$

$$C(s) = \frac{1}{s} - \left[\frac{s + \xi\omega_n}{(s + \xi\omega_n)^2 + \omega_n^2(1 - \xi^2)} + \frac{\xi\omega_n}{(s + \xi\omega_n)^2 + \omega_n^2(1 - \xi^2)} \right]$$

But we have

$$\omega_d = \omega_n \sqrt{1 - \xi^2}$$

$$\omega_d^2 = \omega_n^2(1 - \xi^2)$$

$$= \frac{1}{s} - \left[\frac{s + \xi\omega_n}{(s + \xi\omega_n)^2 + \omega_d^2} + \frac{\xi\omega_n \cdot \omega_d}{\omega_d \left[(s + \xi\omega_n)^2 + \omega_d^2 \right]} \right]$$

$$C(s) = \frac{1}{s} - \left[\frac{s + \xi\omega_n}{(s + \xi\omega_n)^2 + \omega_d^2} + \frac{\xi}{\sqrt{1 - \xi^2}} \frac{\omega_d}{(s + \xi\omega_n)^2 + \omega_d^2} \right]$$

Apply I. L. T

$$C(t) = 1 - \left[e^{-\xi\omega_n t} \cos \omega_d t + \frac{\xi}{\sqrt{1 - \xi^2}} e^{-\xi\omega_n t} \sin \omega_d t \right]$$

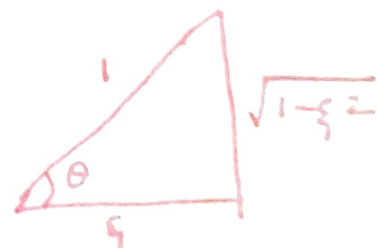
$$\mathcal{L}^{-1} \left\{ \frac{s+a}{(s+a)^2 + b^2} \right\} = e^{-at} \cos bt \quad \mathcal{L}^{-1} \left\{ \frac{b}{(s+a)^2 + b^2} \right\} = e^{-at} \sin bt$$

$$C(t) = 1 - \frac{e^{-\xi\omega_n t}}{\sqrt{1 - \xi^2}} \left[\sqrt{1 - \xi^2} \cos \omega_d t + \xi \sin \omega_d t \right]$$

$$C(t) = 1 - \frac{e^{-\xi\omega_n t}}{\sqrt{1 - \xi^2}} \left[\sin \theta \cos \omega_d t + \cos \theta \sin \omega_d t \right]$$

$$C(t) = 1 - \frac{e^{-\xi\omega_n t}}{\sqrt{1 - \xi^2}} \sin(\omega_d t + \theta)$$

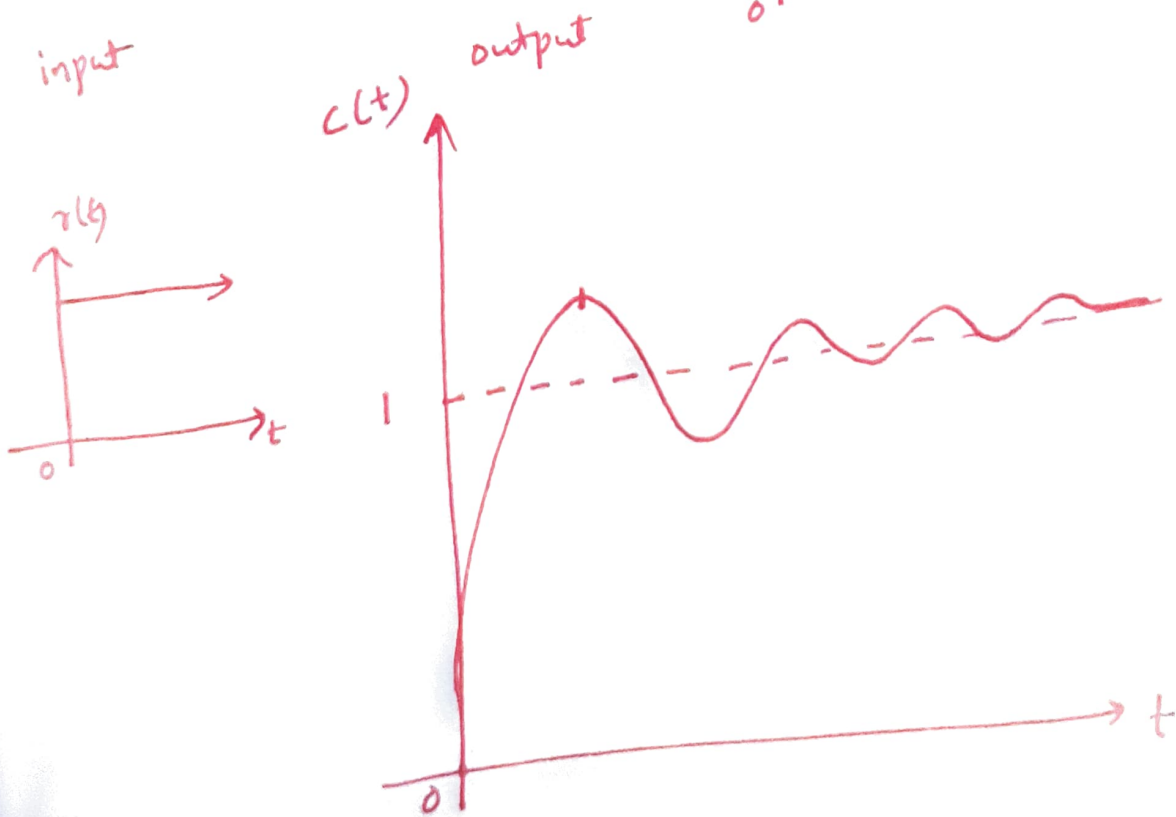
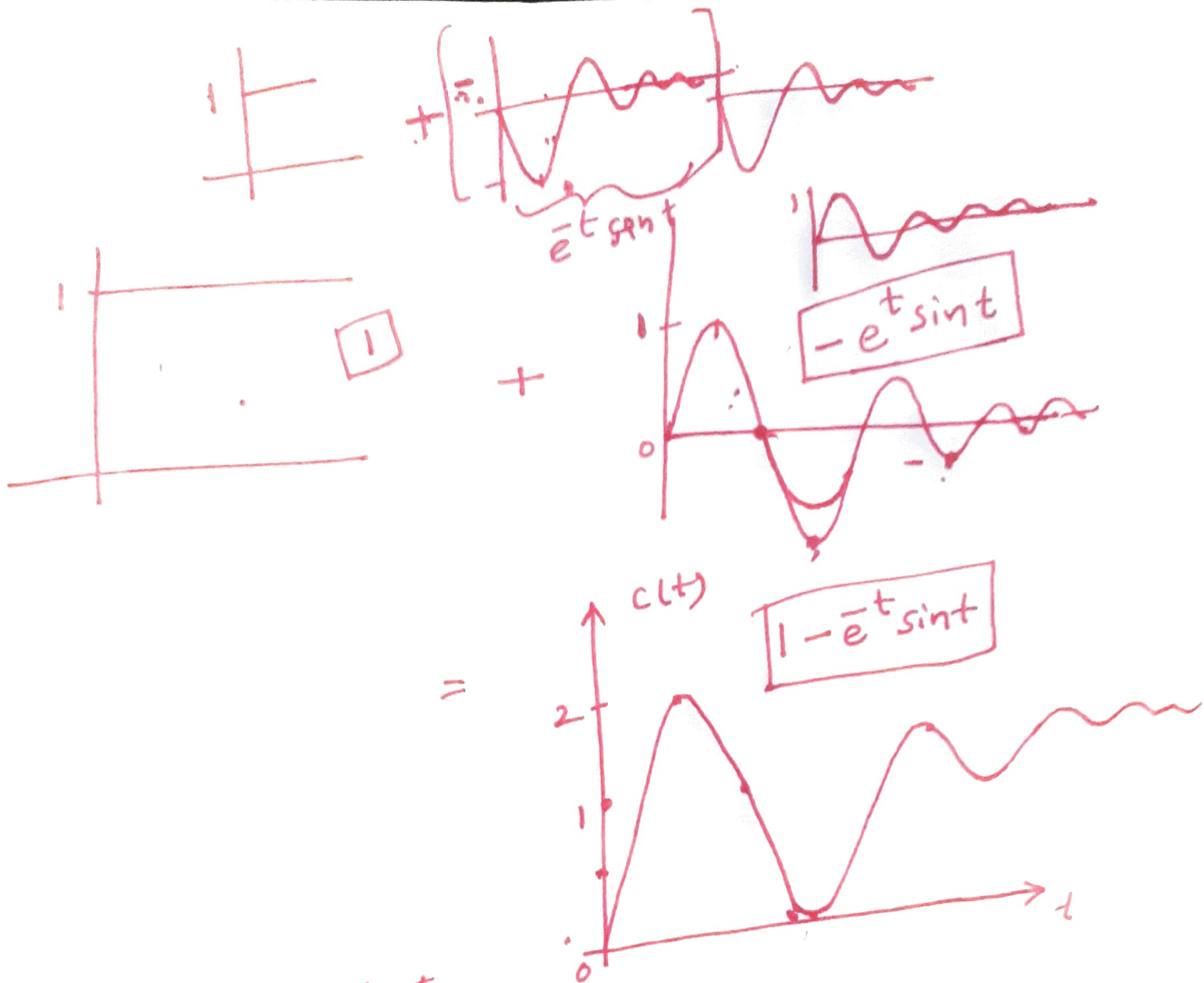
where $\theta = \tan^{-1} \left[\frac{\sqrt{1 - \xi^2}}{\xi} \right]$



$$\cos \theta = \xi$$

$$\sin \theta = \sqrt{1 - \xi^2}$$

$$\tan \theta = \frac{\sqrt{1 - \xi^2}}{\xi}$$



Decreasing amplitude frequency of oscillatory

Response of Critically damped system for unit
step input:

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\xi\omega_n s + \omega_n^2}$$

For critically damping

$$\xi = 1$$

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\omega_n s + \omega_n^2}$$

$$C(s) = \frac{\omega_n^2}{(s + \omega_n)^2} \cdot R(s)$$

But input is unit step

$$r(t) = u(t)$$

$$R(s) = 1/s$$

$$C(s) = \frac{\omega_n^2}{(s + \omega_n)^2} \cdot \frac{1}{s}$$

$$= \frac{\omega_n^2}{s(s + \omega_n)^2} = \frac{A}{s} + \frac{B}{s + \omega_n} + \frac{C}{(s + \omega_n)^2}$$

$$\Rightarrow \omega_n^2 = A(s + \omega_n)^2 + B(s + \omega_n)s + C \cdot s$$

when $s = 0 \Rightarrow \boxed{A = 1}$

$s = -\omega_n \Rightarrow \boxed{C = -\omega_n^2}$

s^2 coeff.

$$A + B = 0$$

$$\boxed{B = -1}$$

$$C(s) = \frac{1}{s} - \frac{1}{s + \omega_n} - \frac{\omega_n}{(s + \omega_n)^2}$$

Apply I.L.T

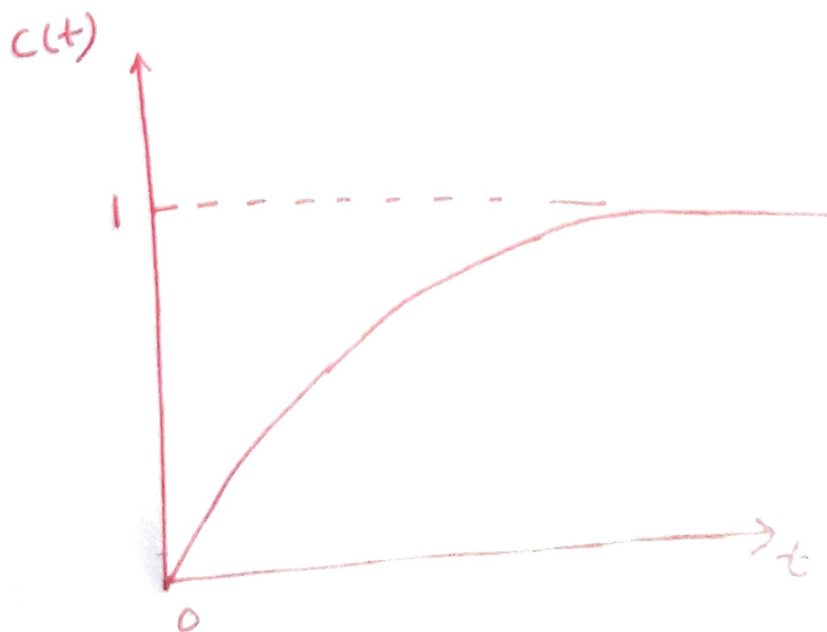
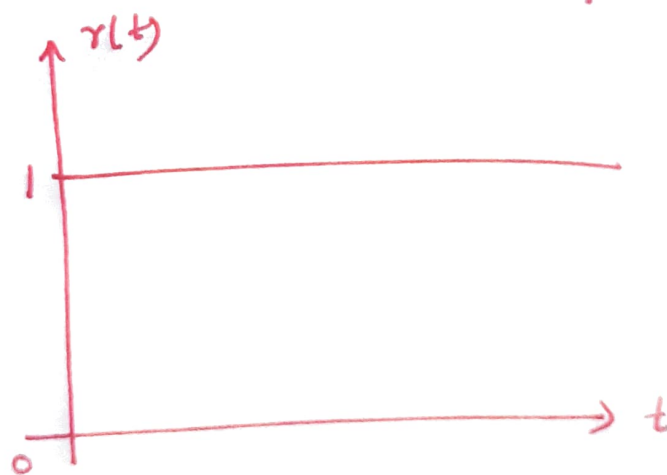
$$\mathcal{L}^{-1}\left\{\frac{1}{s+a}\right\} = e^{-at}$$

$$\mathcal{L}^{-1}\left\{\frac{1}{(s+a)^2}\right\} = t \cdot e^{-at}$$

$$c(t) = 1 - e^{-\omega_n t} - \omega_n t \cdot e^{-\omega_n t}$$

$$c(t) = 1 - e^{-\omega_n t} (1 + \omega_n t)$$

|| - ||



It has no oscillations

Time Response of over damped system for unit step input:

consider standard closed loop second order system

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\xi\omega_n s + \omega_n^2}$$

For overdamped system, $\xi > 1$

The roots are real and unequal.

$$\text{Let } s_1, s_2 = -\xi\omega_n \pm \omega_n \sqrt{\xi^2 - 1}$$

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\xi\omega_n s + \omega_n^2} = \frac{\omega_n^2}{s(s+s_1)(s+s_2)}$$

But here i/p is unit step

$$r(t) = u(t)$$

$$R(s) = 1/s$$

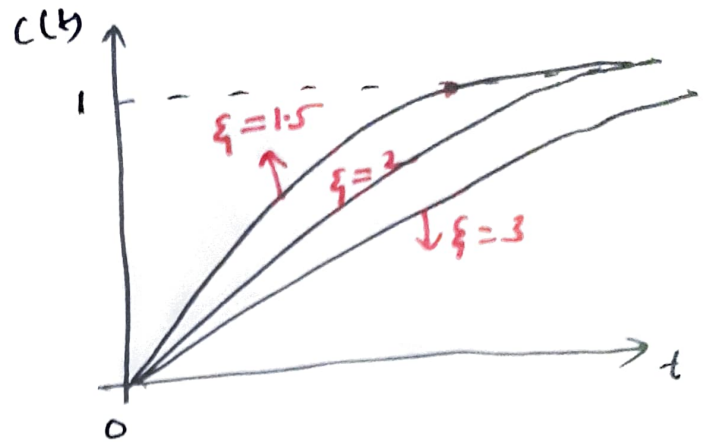
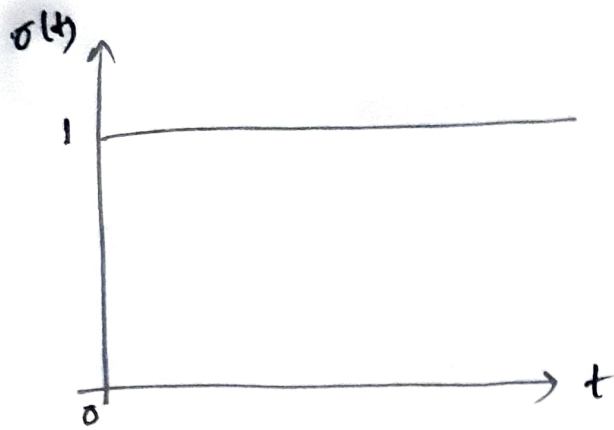
$$C(s) = \frac{\omega_n^2}{s(s+s_1)(s+s_2)}$$

$$C(s) = \frac{A}{s} + \frac{B}{s+s_1} + \frac{C}{s+s_2}$$

Apply I.L.T

$$c(t) = A + B \cdot e^{-s_1 t} + C \cdot e^{-s_2 t}$$

$$C(t) = 1 - \frac{\omega_n}{2\sqrt{\xi^2 - 1}} \left[\frac{e^{-s_1 t}}{s_1} - \frac{e^{-s_2 t}}{s_2} \right]$$

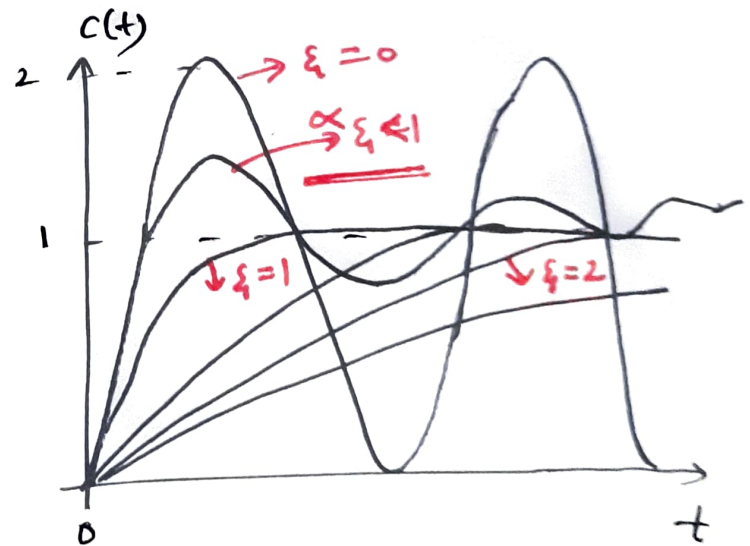
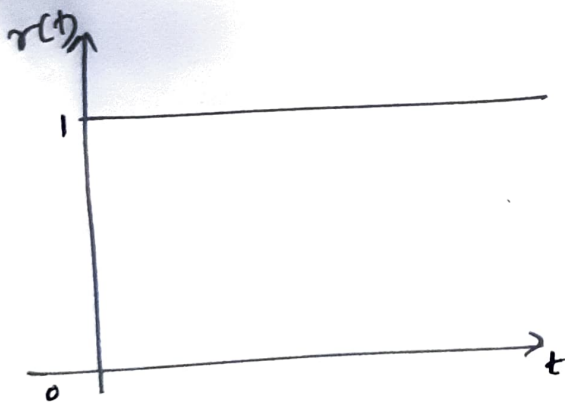


Time Domain Specifications:-

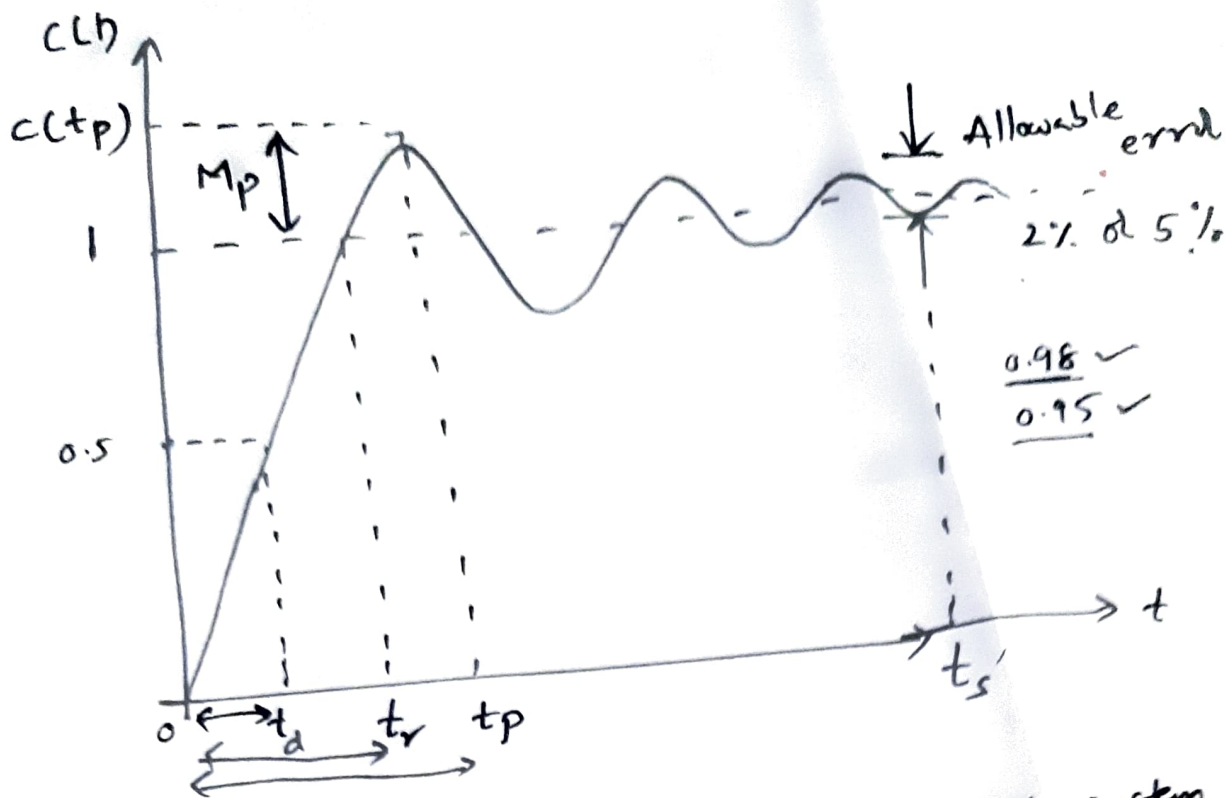


The desired performance characteristics of

control systems.



The transient response of a practical control system often exhibits damped oscillations before reaching steady state.



Response of second order underdamped system to unit step i/p.

- 1) Delay time (t_d)
- 2) Rise time (t_r)
- 3) peak time (t_p)
- 4) Maximum overshoot (M_p)
- 5) settling time (t_s)



10% to 90%

1) Delay time: It is the time taken by the response to reach 50% of its final value. (t_d)

2) Rise time:

underdamped $\rightarrow$ " " 0% to 100%

critically damped $\rightarrow$ " " 10% to 90%

over damped $\rightarrow$ " " 5% to 95%

peak time (t_p): It is the time taken by the response to reach peak value in the very first time.

peak overshoot (M_p): It is ratio of max. peak value to the final value.

Let ~~$c(t)$~~ $c(\infty)$ = Final value of $c(t)$

$c(t_p)$ = Max. value of $c(t)$

$$\text{peak overshoot } M_p = \frac{c(t_p) - c(\infty)}{c(\infty)}$$

$$\% \text{ peak overshoot } (\% M_p) = \frac{c(t_p) - c(\infty)}{c(\infty)} \times 100$$

Settling time (t_s): It is defined as the time taken by the response to reach and stay within a specified error. It is expressed in terms of final value.
(2% or 5%).

Expressions for Time domain specifications:

Rise time (t_r):

The unit step response of second order system for underdamped case is given by

$$c(t) = 1 - \frac{e^{-\xi \omega_n t}}{\sqrt{1-\xi^2}} \sin(\omega_d t + \theta)$$

$$\text{At } t = t_r, \quad c(t) = c(t_r) = 1$$

$$c(t_r) = 1 - \frac{e^{-\xi \omega_n t_r}}{\sqrt{1-\xi^2}} \sin(\omega_d t_r + \theta) = 1$$

$$- \frac{e^{-\xi \omega_n t_r}}{\sqrt{1-\xi^2}} \sin(\omega_d t_r + \theta) = 0$$

Since $e^{-\xi \omega_n t_r} \neq 0$, the term $\sin(\omega_d t_r + \theta) = 0$
[when $\theta = 0, \pi, 2\pi, \dots \Rightarrow \sin \theta = 0$]

$$\omega_d t_r + \theta = \pi$$

$$\omega_d t_r = \pi - \theta$$

$$\boxed{t_r = \frac{\pi - \theta}{\omega_d}}$$

$$\text{where } \omega_d = \omega_n \sqrt{1-\xi^2}$$

$$\theta = \tan^{-1}\left(\frac{\sqrt{1-\xi^2}}{\xi}\right)$$

peak time (t_p):-

To find the expression for t_p , differentiate $c(t)$ w.r.t t and equate to zero.

$$\text{i.e. } \left. \frac{d}{dt} c(t) \right|_{t=t_p} = 0$$

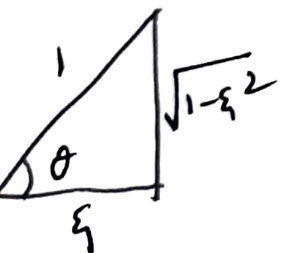
$$\text{But } c(t) = 1 - \frac{e^{-\xi \omega_n t}}{\sqrt{1-\xi^2}} \cdot \sin(\omega_d t + \theta)$$

Diff. $c(t)$ w.r.t t

$$\frac{d}{dt} c(t) = -\frac{e^{-\xi \omega_n t}}{\sqrt{1-\xi^2}} \cdot \cos(\omega_d t + \theta) \omega_d - \frac{e^{-\xi \omega_n t}}{\sqrt{1-\xi^2}} (-\xi \omega_n) \sin(\omega_d t + \theta)$$

$$\text{But } \omega_d = \omega_n \sqrt{1-\xi^2}$$

$$\therefore \frac{d}{dt} c(t) = \frac{e^{-\xi \omega_n t}}{\sqrt{1-\xi^2}} (\xi \omega_n) \sin(\omega_d t + \theta) - \frac{\omega_n \sqrt{1-\xi^2}}{\sqrt{1-\xi^2}} \cdot e^{-\xi \omega_n t} \cos(\omega_d t + \theta)$$



$$\begin{aligned} &= \frac{\omega_n e^{-\xi \omega_n t}}{\sqrt{1-\xi^2}} \left(\xi \sin(\omega_d t + \theta) - \sqrt{1-\xi^2} \cos(\omega_d t + \theta) \right) \\ &= \frac{\omega_n e^{-\xi \omega_n t}}{\sqrt{1-\xi^2}} \left(\cos \theta \cdot \sin(\omega_d t + \theta) - \sin \theta \cos(\omega_d t + \theta) \right) \\ &= \frac{\omega_n}{\sqrt{1-\xi^2}} \cdot e^{-\xi \omega_n t} \sin \omega_d t \end{aligned}$$

at $t = t_p$, $\frac{d}{dt}[C(t)] = 0$

$$\therefore \frac{\omega_n}{\sqrt{1-\xi^2}} e^{-\xi\omega_n t_p} \sin(\omega_d t_p) = 0$$

Since $e^{-\xi\omega_n t} \neq 0$, the term $\sin(\omega_d t_p) = 0$

When $\theta = 0, \pi, 2\pi, \dots \Rightarrow \sin\theta = 0$

$$\sin(\omega_d t_p) = 0$$

$$\omega_d t_p = \pi$$

peak time $\boxed{t_p = \frac{\pi}{\omega_d}}$

where $\omega_d = \omega_n \sqrt{1-\xi^2}$

$$\boxed{t_p = \frac{\pi}{\omega_n \sqrt{1-\xi^2}}}$$

peak overshoot (M_p) :-

$$\% M_p = \frac{c(t_p) - c(\infty)}{c(\infty)} \times 100$$

where $c(t_p)$ = peak response ($t = t_p$)

$c(\infty)$ = Final steady state value.

The unit step response of second order system in under damped case is given by

$$c(t) = 1 - \frac{e^{-\xi \omega_n t} \sin(\omega_d t + \theta)}{\sqrt{1 - \xi^2}}$$

$$\text{At } t = \infty, \quad c(t) = c(\infty) = 1 - \frac{e^{-\infty} \sin(\omega_d(\infty) + \theta)}{\sqrt{1 - \xi^2}} \\ = 1 - 0 = 1$$

$$\text{At } t = t_p, \quad c(t) = c(t_p) = 1 - \frac{e^{-\xi \omega_n t_p} \sin(\omega_d t_p + \theta)}{\sqrt{1 - \xi^2}}$$

$$\text{But } t_p = \frac{\pi}{\omega_d} = \frac{\pi}{\omega_n \sqrt{1 - \xi^2}}$$

$$= 1 - \frac{e^{-\xi \omega_n \cdot \frac{\pi}{\omega_n \sqrt{1 - \xi^2}}} \sin(\pi + \theta)}{\sqrt{1 - \xi^2}} = 1 + \frac{e^{-\xi \pi / \sqrt{1 - \xi^2}} \sin \theta}{\sqrt{1 - \xi^2}} \\ = 1 + e^{-\xi \pi / \sqrt{1 - \xi^2}}$$

$$\% \text{ peak overshoot } (\% M_p) = \frac{c(t_p) - c(\infty)}{c(\infty)} \times 100$$

$$= \frac{1 + e^{-\xi \pi / \sqrt{1 - \xi^2}} - 1}{1} \times 100 \\ \% M_p = e^{-\xi \pi / \sqrt{1 - \xi^2}} \times 100$$

Settling time (t_s):-

The response of second order system has two components. They are

1) Decaying Exponential component $\left(\frac{e^{-\xi \omega_n t}}{\sqrt{1-\xi^2}} \right) \Delta$

2) Sinusoidal component $[\sin(\omega_d t + \theta)] \Delta$

In this, decaying exponential component reduces the oscillations produced by sinusoidal component. Hence the settling time is decided by exponential component.

For 2% tolerance error, at $t = t_s$

$$\frac{e^{-\xi \omega_n t_s}}{\sqrt{1-\xi^2}} = 0.02$$

For least value of ξ , $e^{-\xi \omega_n t_s} = 0.02$

$$-\xi \omega_n t_s = \ln(0.02)$$

$$-\xi \omega_n t_s = -4$$

$$\boxed{t_s = \frac{4}{\xi \omega_n}} \quad \text{for 2\%}$$

For 5% tolerance error, at $t = t_s$

$$\frac{e^{-\xi \omega_n t_s}}{\sqrt{1-\xi^2}} = 0.05$$

$$e^{-\xi \omega_n t_s} = 0.05$$

$$-\xi \omega_n t_s = \ln(0.05)$$

$$-\xi \omega_n t_s = -3$$

$$\boxed{t_s = \frac{3}{\xi \omega_n}} \quad \text{for 5\%}$$

Rise time $t_r = \frac{\pi - \theta}{\omega_d}$

where $\theta = \tan^{-1}\left(\frac{\sqrt{1-\xi^2}}{\xi}\right)$

$$\omega_d = \omega_n \sqrt{1-\xi^2}$$

peak time $t_p = \frac{\pi}{\omega_d}$

where $\omega_d = \omega_n \sqrt{1-\xi^2}$

% peak overshoot $\%M_p = e^{-\xi\pi/\sqrt{1-\xi^2}} \times 100$

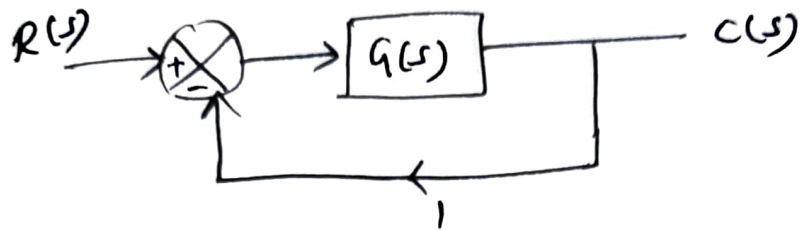
settling time $t_s = \frac{4}{\xi\omega_n}$, for 2%
 $= \frac{3}{\xi\omega_n}$, for 5%

① obtain the response of unity f/b system

where open loop Transfer function (OLTF) is $G(s) = \frac{4}{s(s+5)}$

and where the i/p is unit step.

Sol:



The closed loop Transfer function (CLTF)

$$\frac{C(s)}{R(s)} = \frac{G(s)}{1+G(s)} \quad [H(s) = 1]$$

$$\frac{C(s)}{R(s)} = \frac{\frac{4}{s(s+5)}}{1 + \frac{4}{s(s+5)}} = \frac{\frac{4}{s(s+5)}}{\frac{s(s+5) + 4}{s(s+5)}}$$

$$\frac{C(s)}{R(s)} = \frac{4}{s^2 + 5s + 4} = \frac{4}{(s+1)(s+4)}$$

$$C(s) = \frac{4}{(s+1)(s+4)} \cdot R(s)$$

But input is unit step i.e. $r(t) = u(t)$
 $R(s) = \frac{1}{s}$

$$C(s) = \frac{4}{s(s+1)(s+4)}$$

$$= \frac{A}{s} + \frac{B}{s+1} + \frac{C}{s+4}$$

$$4 = A(s+1)(s+4) + Bs(s+4) + Cs(s+1)$$

$$\text{If } s=0 \Rightarrow A = 1$$

$$s=-1 \Rightarrow B = -4/3$$

$$s=-4 \Rightarrow C = 1/3$$

$$C(s) = \frac{1}{s} - \frac{4/3}{s+1} + \frac{1/3}{s+4}$$

Apply Inverse L.T

$$c(t) = 1 - \frac{4}{3} e^{-t} + \frac{1}{3} e^{-4t}$$

2) The response of servomechanism is $c(t) = 1 + 0.2 e^{-60t} - 1.2 e^{-10t}$ when subjected to unit step input, obtain CLTF and Determine undamped natural frequency, damping Ratio.

Sol.

$$\text{Given } c(t) = 1 + 0.2 e^{-60t} - 1.2 e^{-10t}$$

Apply L.T

$$C(s) = \frac{1}{s} + 0.2 \left[\frac{1}{s+60} \right] - 1.2 \left[\frac{1}{s+10} \right]$$

$$= \frac{(s+60)(s+10) + 0.2s(s+10) - 1.2s(s+60)}{s(s+60)(s+10)}$$

$$= \frac{s^2 + 70s + 600 + 0.2s^2 + 2s - 1.2s^2 - 72s}{s(s+60)(s+10)}$$

$$C(s) = \frac{600}{s(s+60)(s+10)}$$

$$= \frac{1}{s} \cdot \frac{600}{(s+60)(s+10)}$$

Since input is unit step, $R(s) = 1/s$

$$C(s) = R(s) \cdot \frac{600}{(s+60)(s+10)}$$

$$\frac{C(s)}{R(s)} = \frac{600}{(s+60)(s+10)}$$

$$\text{CLTF} \Rightarrow \boxed{\frac{C(s)}{R(s)} = \frac{600}{s^2 + 70s + 600}}$$

damping Ratio & undamped natural freq. can be estimated by comparing CLTF with standard second order system.

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\xi\omega_n s + \omega_n^2} = \frac{600}{s^2 + 70s + 600}$$

$$\omega_n^2 = 600$$

$$\omega_n = \sqrt{600}$$

$$\boxed{\omega_n = 24.49 \text{ rad/sec}}$$

$$2\xi\omega_n = 70$$

$$2\xi(24.49) = 70$$

$$\xi = \frac{70}{2 \times 24.49}$$

$$\boxed{\xi = 1.43}$$

3) The unity f/b system characterized by an OLTF

$$G(s) = \frac{k}{s(s+10)} \quad \text{Determine the gain 'k' so that}$$

the system will have a damping ratio of 0.5. For this

value of 'k', Determine settling time, peak time

and peak overshoot.

Sol:

$$\text{OLTF} = \frac{G(s)}{1+G(s)} = \frac{k}{s(s+10)}$$

$$\begin{aligned} \text{CLTF} &= \frac{G(s)}{1+G(s)} \\ &= \frac{\frac{k}{s(s+10)}}{1 + \frac{k}{s(s+10)}} = \frac{k}{s^2 + 10s + k} \end{aligned}$$

compare it with

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\xi\omega_n s + \omega_n^2}$$

$$\omega_n^2 = k$$

$$\omega_n = \sqrt{k}$$

$$2\xi\omega_n = 10$$

$$2(0.5)(\sqrt{k}) = 10$$

$$\sqrt{k} = 10$$

$$\boxed{k = 100} \quad \checkmark$$

$$\xi = 0.5$$

$$\omega_n = \sqrt{k} = 10 \text{ rad/sec} \quad \checkmark$$

$$\begin{aligned} \% m_p &= e^{-\xi\pi/\sqrt{1-\xi^2}} \times 100 = e^{-0.5\pi/\sqrt{1-(0.5)^2}} \times 100 \\ &= 16.3\% \end{aligned}$$

$$t_p = \frac{\pi}{\omega_d} = \frac{\pi}{\omega_n \sqrt{1-\xi^2}} = \frac{\pi}{10 \sqrt{1-0.25}} = 0.363 \text{ sec}$$

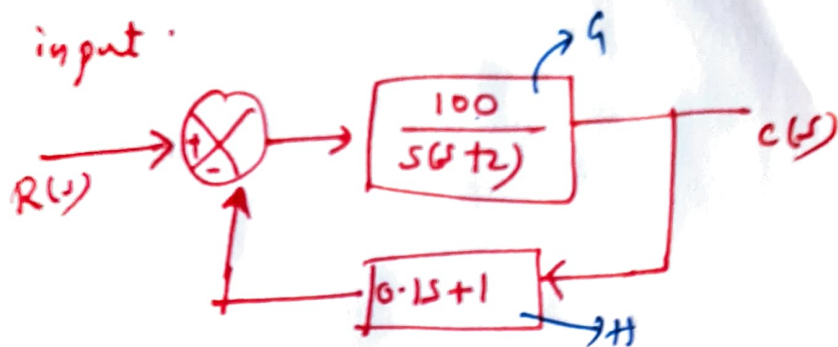
$$t_s = \frac{4}{\xi \omega_n} \quad \text{for } 2\%.$$

$$= \frac{4}{0.5(10)} = \frac{4}{5} = 0.8 \text{ sec}$$

$$t_s = \frac{3}{\xi \omega_n} \quad \text{for } 5\%.$$

$$= \frac{3}{0.5(10)} = \frac{3}{5} = 0.6 \text{ sec}$$

4) A positional control system with velocity feedback is shown in figure. Find the response of the system to a step input.



Sol Time Response ? $\rightarrow C(t)$

$$\left(\frac{C(s)}{R(s)} \right) \Rightarrow R(s) = \frac{1}{s}$$

Apply I.L.T
 $C(t)$

$$\text{CLTF} \quad \frac{C(s)}{R(s)} = \frac{G(s)}{1 + G(s)H(s)}$$

$$= \frac{\frac{100}{s(s+2)}}{1 + \frac{100}{s(s+2)} \cdot (0.1s+1)}$$

$$= \frac{100}{s^2 + 2s + 10s + 100}$$

$$\frac{C(s)}{R(s)} = \frac{100}{s^2 + 12s + 100}$$

Given input $r(t) = u(t)$
 $R(s) = \frac{1}{s}$

$$C(s) = \frac{100}{s(s^2 + 12s + 100)} = \frac{A}{s} + \frac{Bs+C}{s^2 + 12s + 100}$$

$$\Rightarrow 100 = A(s^2 + 12s + 100) + (Bs+C)s$$

When $s=0 \Rightarrow 100 = 100A \Rightarrow \boxed{A=1}$

compare s^2 coefficients

$$A + B = 0$$

$$\Rightarrow A = -B \Rightarrow \boxed{B = -1}$$

compare s coefficients

$$12A + C = 0 \Rightarrow \boxed{C = -12A}$$
$$\boxed{C = -12}$$

$$C(s) = \frac{1}{s} + \frac{-s-12}{s^2+12s+100}$$

$$= \frac{1}{s} - \frac{s+12}{s^2+12s+100}$$

$$= \frac{1}{s} - \frac{(s+6)+6}{s^2+12s+36+64}$$

$$= \frac{1}{s} - \frac{s+6}{(s+6)^2+8^2} - \frac{6}{(s+6)^2+8^2}$$

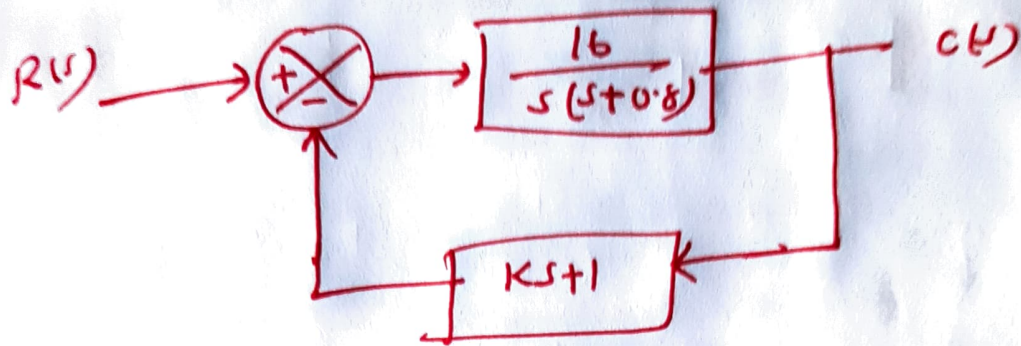
$$= \frac{1}{s} - \frac{s+6}{(s+6)^2+8^2} - \frac{6}{8} \frac{8}{(s+6)^2+8^2}$$

$$= \frac{1}{s} - \frac{e^{-6t}}{8} \cos 8t - \frac{6}{8} e^{-6t} \sin 8t$$

$$\boxed{C(t) = \frac{1}{s} - e^{-6t} \left[\frac{6}{8} \sin 8t + \cos 8t \right]}$$

$$\left\{ \begin{array}{l} \frac{s+6}{(s+6)^2+6^2} \\ \frac{6}{(s+6)^2+6^2} \end{array} \right.$$

→ A positional control system with velocity $\frac{lb}{s}$ is shown in figure. Given that $\xi = 0.5$, find time response of the system and $t_r, t_p, \%MP, t_s$



$$K = 0.2$$

$$\omega_n = 4 \text{ rad/sec}$$

$$c(t) = 1 - e^{-2t} \left[\frac{1}{\sqrt{3}} \sin \sqrt{12}t + \cos \sqrt{12}t \right]$$

$$\text{Rise time } t_r = 0.6046 \text{ sec}$$

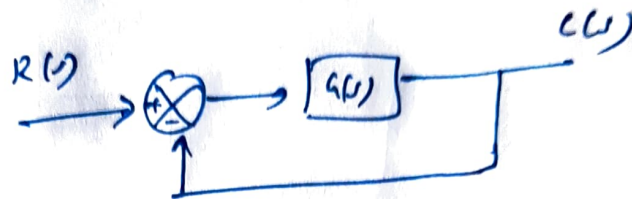
$$t_p = 0.907 \text{ sec}$$

$$\%MP = 16.3\%$$

$$t_s = 1.5 \text{ sec} \quad 2\% \\ = 2 \text{ sec} \quad 5\%$$

The OLTF of a unity f/b system is $G(s) = \frac{K}{s(sT+1)}$ where K, T are positive constants. By what factor should the amplifier gain ' K ' be reduced, so that peak overshoot of system is reduced from 75% to 25%.

Sol:



$$\begin{aligned} \text{CLTF } \frac{C(s)}{R(s)} &= \frac{G(s)}{1+G(s)} \\ &= \frac{\frac{K}{s(sT+1)}}{1 + \frac{K}{s(sT+1)}} \end{aligned}$$

$$\boxed{\frac{C(s)}{R(s)} = \frac{K}{s^2T + s + K}}$$

To find ξ, ω_n we have to compare above equation with standard second order T.F. i.e.

$$\boxed{\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\xi\omega_n s + \omega_n^2}}$$

$$\omega_n^2 = K \quad \frac{C(s)}{R(s)} = \frac{K/T}{s^2 + s/T + K/T}$$

$$\begin{aligned} \omega_n^2 &= K/T \\ \omega_n &= \sqrt{K/T} \end{aligned}$$

$$\begin{aligned} 2\xi\omega_n &= 1/T \\ 2\xi\sqrt{K/T} &= 1/T \\ \xi &= \frac{1}{2\sqrt{KT}} \end{aligned}$$

when $m_p = 0.75$, Let $\xi = \xi_1$ and $k = k_1$,

when $m_p = 0.25$, Let $\xi = \xi_2$ and $k = k_2$

when $m_p = 0.75$

$$\% M_p = \frac{-\xi_1 \pi / \sqrt{1-\xi_1^2}}{e} \times 100$$

$$75 = \frac{-\xi_1 \pi / \sqrt{1-\xi_1^2}}{e} \times 100$$

$$\frac{-\xi_1 \pi / \sqrt{1-\xi_1^2}}{e} = 0.75$$

$$-\xi_1 \pi / \sqrt{1-\xi_1^2} = \ln(0.75)$$

$$\xi_1 \pi / \sqrt{1-\xi_1^2} = 0.28$$

$$\xi_1 \pi = 0.28 \sqrt{1-\xi_1^2}$$

$$\xi_1^2 \pi^2 = (0.28)^2 (1-\xi_1^2)$$

$$\xi = \frac{1}{2\sqrt{kT}}$$

$$\frac{\pi^2}{4k_1 T} = (0.28)^2 \left(1 - \frac{1}{4k_1 T}\right)$$

$$\boxed{k_1 = \frac{30.06}{T}}$$

$m_p = 0.25$

$$20.25 = \frac{-\xi_2 \pi / \sqrt{1-\xi_2^2}}{e}$$

$$\frac{-\xi_2 \pi}{\sqrt{1-\xi_2^2}} = \ln(0.25)$$

$$\xi_2 \pi = \ln(0.25) \sqrt{1-\xi_2^2}$$

1,

1,

$$\boxed{k_2 = \frac{1.53}{T}}$$

$$\frac{k_1}{k_2} = \frac{\frac{1}{T} (30.06)}{\frac{1}{T} (1.53)} = 19.6$$

$$\boxed{k_1 = 19.6 k_2}$$

$$\Delta \quad \boxed{k_2 = \frac{1}{19.6} k_1}$$

To reduce peak overshoot from 0.75 to 0.25
 k should be reduced by 19.6 times.

Type Number of control systems:-

The type number is specified for loop transfer function $G(s) \cdot H(s)$. The number poles of loop T.F. lying at origin decides the type-number of the system.

$$\begin{aligned} \text{consider } G(s)H(s) &= K \cdot \frac{P(s)}{Q(s)} \\ &= K \frac{(s+z_1)(s+z_2) \dots}{s^N (s+p_1)(s+p_2) \dots} \end{aligned}$$

where $z_1, z_2 \dots$ - zeros of the T.F

$p_1, p_2 \dots$ - poles.

N = Number of poles at origin.

If $N=0$, then the system is type-0 system

$N=1$, " type-1

$N=2$ " type-2

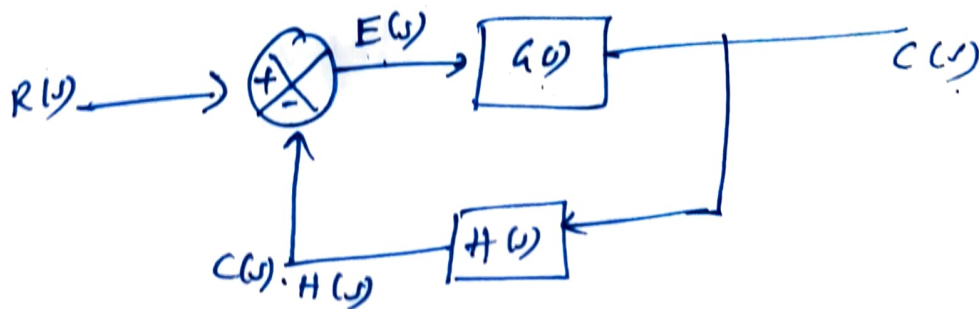
$N=3$ " type-3

and so on

Steady state error:-

It is the value of error signal $e(t)$, when t tends to infinity. The steady state error is a measure of system accuracy.

Consider a closed loop system



Let $R(s)$ = input signal

$C(s)$ = output

$E(s)$ = error signal

$C(s) \cdot H(s)$ = Feedback signal.

The error signal $E(s) = R(s) - C(s) \cdot H(s) \rightarrow (1)$

But $C(s) = E(s) \cdot G(s) \rightarrow (2)$

Sub (2) in (1)

$$E(s) = R(s) - E(s) \cdot G(s) \cdot H(s)$$

$$\boxed{-E(s) = \frac{R(s)}{1 + G(s) \cdot H(s)}} \rightarrow (3)$$

Let $e(t)$ = error signal in time domain

$$e(t) = \mathcal{L}^{-1}\{E(s)\} = \mathcal{L}^{-1}\left\{\frac{R(s)}{1+G(s)H(s)}\right\}$$

Let e_{ss} = steady state error.
↓ error signal at $t \rightarrow \infty$

$$e_{ss} = \lim_{t \rightarrow \infty} e(t)$$

The final value Theorem of L.T. states that

If $X(s) = \mathcal{L}\{x(t)\}$ then $\lim_{t \rightarrow \infty} x(t) = \lim_{s \rightarrow 0} s \cdot X(s)$

$$\lim_{t \rightarrow \infty} e(t) = \lim_{s \rightarrow 0} s \cdot E(s).$$

$$e_{ss} = \lim_{s \rightarrow 0} \frac{s \cdot R(s)}{1+G(s)H(s)}$$

where $R(s) = \mathcal{L}\{r(t)\}$

$$G(s) \cdot H(s) = \text{loop T.F.}$$

Static error constants:

- i) positional error constant $K_p = \lim_{s \rightarrow 0} G(s) \cdot H(s)$
- ii) velocity error " $K_v = \lim_{s \rightarrow 0} s \cdot G(s) \cdot H(s)$
- iii) Acceleration " $K_a = \lim_{s \rightarrow 0} s^2 G(s) \cdot H(s)$

Steady state error when the i/p is unit step

$$\text{steady state error } e_{ss} = \lim_{s \rightarrow 0} \frac{s \cdot R(s)}{1 + G(s) \cdot H(s)}$$

when the input is unit step

$$R(s) = 1/s$$

$$\begin{aligned} e_{ss} &= \lim_{s \rightarrow 0} \frac{s \cdot 1/s}{1 + G(s) \cdot H(s)} \\ &= \frac{1}{1 + \underbrace{\lim_{s \rightarrow 0} G(s) \cdot H(s)}} \end{aligned}$$

$$e_{ss} = \frac{1}{1 + K_p}$$

$$\text{where } K_p = \lim_{s \rightarrow 0} G(s) \cdot H(s)$$

Type-0 system

$$K_p = \lim_{s \rightarrow 0} G(s)H(s) = \lim_{s \rightarrow 0} K \frac{(s+z_1)(s+z_2) \dots}{s^0(s+p_1)(s+p_2) \dots}$$
$$= \frac{K \cdot z_1 \cdot z_2 \dots}{p_1 \cdot p_2} = \text{constant}$$

$$e_{ss} = \frac{1}{1+K_p} = \text{constant}$$

Hence in type 0 system when the i/p is unit step there will be a constant e_{ss} .

Type-1 system

$$K_p = \lim_{s \rightarrow 0} G(s)H(s) = \lim_{s \rightarrow 0} K \frac{(s+z_1)(s+z_2) \dots}{s^1(s+p_1)(s+p_2) \dots}$$
$$= \frac{K}{0} = \infty$$

$$e_{ss} = \frac{1}{1+K_p} = \frac{1}{\infty} = 0$$

In system, with type number ≥ 1 for unit step

i/p, $\boxed{e_{ss} = 0}$

steady state error when the i/p is Ramp ^{unit} '1/s' :-

$$\text{steady state error } e_{ss} = \lim_{s \rightarrow 0} \frac{s \cdot R(s)}{1 + G(s)H(s)}$$

when the i/p is unit ramp

$$R(s) = 1/s^2$$

$$\therefore e_{ss} = \lim_{s \rightarrow 0} \frac{s \cdot 1/s^2}{1 + G(s)H(s)} = \lim_{s \rightarrow 0} \frac{1}{s(1 + G(s)H(s))}$$

$$= \lim_{s \rightarrow 0} \frac{1}{s + s \cdot G(s)H(s)}$$

$$e_{ss} = \frac{1}{\lim_{s \rightarrow 0} s \cdot G(s)H(s)} = \frac{1}{K_V}$$

↓
velocity error constant

where $K_V = \lim_{s \rightarrow 0} s \cdot G(s)H(s)$

Type-0 system:-

$$G(s)H(s) = \frac{K(s+z_1)(s+z_2) \dots}{(s+p_1)(s+p_2) \dots}$$

$$K_V = \lim_{s \rightarrow 0} s \cdot G(s)H(s) = \lim_{s \rightarrow 0} \frac{s \cdot K(s+z_1)(s+z_2) \dots}{(s+p_1)(s+p_2) \dots} = 0$$

$$e_{ss} = 1/K_V = 1/0 = \infty$$

Type-1 system:-

$$K_V = \lim_{s \rightarrow 0} s \cdot G(s)H(s) = \lim_{s \rightarrow 0} \frac{s \cdot K(s+z_1)(s+z_2) \dots}{s(s+p_1)(s+p_2) \dots}$$

$$= \frac{K \cdot z_1 \cdot z_2 \dots}{p_1 \cdot p_2 \dots} = \text{constant}$$

$$e_{ss} = \frac{1}{K_V} = \frac{1}{\text{constant}} = \text{constant}$$

Type-2 system:

$$K_v = \lim_{s \rightarrow 0} s \cdot G(s)H(s) = \lim_{s \rightarrow 0} \frac{s \cdot K(s+z_1)(s+z_2) \dots}{s^2(s+z_1)(s+z_2) \dots}$$
$$= \frac{1}{0} = \infty$$

$$e_{ss} = \frac{1}{K_v} = \frac{1}{\infty} = 0$$

In the system with type number 2 and above,

for unit ramp i/p, K_v is ∞ and $e_{ss} = 0$

steady state error when the i/p is unit parabolic i/p:

$$\text{steady state error } e_{ss} = \lim_{s \rightarrow 0} \frac{s \cdot R(s)}{1 + G(s)H(s)}$$

when the i/p is unit parabolic
 $R(s) = 1/s^3$

$$e_{ss} = \lim_{s \rightarrow 0} \frac{s \cdot 1/s^3}{1 + G(s)H(s)}$$

$$= \lim_{s \rightarrow 0} \frac{1}{s^2 + s^2 G(s)H(s)}$$

$$= \cancel{\lim_{s \rightarrow 0} \frac{1}{s^2}} \cdot \lim_{s \rightarrow 0} \frac{1}{s^2 G(s)H(s)}$$

$e_{ss} = 1/K_a$

$$\text{where } K_a = \lim_{s \rightarrow 0} s^2 G(s)H(s)$$

acceleration error constant

Type-0 system:

$$K_a = \lim_{s \rightarrow 0} s^L \cdot G(s) \cdot H(s) = \lim_{s \rightarrow 0} \frac{s^L \cdot K(s+z_1)(s+z_2) \dots}{(s+p_1)(s+p_2) \dots} = 0$$

$$e_{ss} = \frac{1}{K_a} = \frac{1}{0} = \infty$$

Type-1 system:

$$K_a = \lim_{s \rightarrow 0} s^1 \cdot \frac{K(s+z_1)(s+z_2) \dots}{(s+p_1)(s+p_2) \dots} = 0$$

$$e_{ss} = \frac{1}{K_a} = \frac{1}{0} = \infty$$

Type-2 system:

$$K_a = \lim_{s \rightarrow 0} s^2 \cdot \frac{K(s+z_1)(s+z_2) \dots}{s^2(s+p_1)(s+p_2) \dots} = \text{constant}$$

$$e_{ss} = \frac{1}{K_a} = \text{constant}$$

Type-3 system:

$$K_a = \lim_{s \rightarrow 0} s^3 \cdot \frac{K(s+z_1)(s+z_2) \dots}{s^3(s+p_1)(s+p_2) \dots} = \frac{1}{0} = \infty$$

$$e_{ss} = \frac{1}{K_a} = 0$$

In system type number is 3 and above, for unit parabolic i/p

$$K_a = \infty \text{ \& } e_{ss} = 0$$

Static error constants for various type numbers of system

Error constant	Type number			
	0	1	2	3
K_p	constant	∞	∞	∞
K_v	0	constant	∞	∞
K_a	0	0	constant	∞

steady state error for various types of inputs

Input signal	Type-number			
	0	1	2	3
unit step	$\frac{1}{1+K_p}$	0	0	0
unit ramp	∞	$\frac{1}{K_v}$	0	0
unit parabolic	∞	∞	$\frac{1}{K_a}$	0

Formulae to calculate steady state error:

$$K_p = \lim_{s \rightarrow 0} G(s)H(s)$$

$$K_v = \lim_{s \rightarrow 0} s \cdot G(s)H(s)$$

$$K_a = \lim_{s \rightarrow 0} s^2 G(s)H(s)$$

If it is unity feedback

$$= \lim_{s \rightarrow 0} G(s)$$

$$= \lim_{s \rightarrow 0} s \cdot G(s)$$

$$= \lim_{s \rightarrow 0} s^2 G(s)$$

unit step input $r(t) = u(t) / 1$

$$e_{ss} = \frac{1}{1 + K_p}$$

unit ramp input $r(t) = t \cdot u(t) / t$

$$e_{ss} = \frac{1}{K_v}$$

unit parabolic i/p $r(t) = \frac{t^2}{2} u(t) / \frac{t^2}{2}$

$$e_{ss} = \frac{1}{K_a}$$

1) For a unity feedback system the OLTF

$$G(s) = \frac{10(s+2)}{s^2(s+1)}$$

a) Find static error constant.

b) Find steady state error when i/p

$$R(s) = \frac{3}{s} - \frac{2}{s^2} + \frac{1}{3s^3}$$

sol: a) For unity f/b system $H(s) = 1$

$$\text{position error constant } K_p = \lim_{s \rightarrow 0} G(s) \cdot H(s)$$

$$= \lim_{s \rightarrow 0} G(s)$$

$$= \lim_{s \rightarrow 0} \frac{10(s+2)}{s^2(s+1)}$$

$$= \frac{1}{\infty} = 0$$

$$\text{velocity error constant } K_v = \lim_{s \rightarrow 0} s \cdot G(s) \cdot H(s) = \lim_{s \rightarrow 0} s \cdot G(s)$$

$$= \lim_{s \rightarrow 0} \cancel{s} \cdot \frac{10(s+2)}{s^{\cancel{2}}(s+1)} = \frac{1}{0} = \infty$$

$$\text{acceleration error constant } K_a = \lim_{s \rightarrow 0} s^2 G(s) \cdot H(s) = \lim_{s \rightarrow 0} s^2 G(s)$$

$$= \lim_{s \rightarrow 0} \cancel{s^2} \cdot \frac{10(s+2)}{\cancel{s^2}(s+1)}$$

$$= \frac{10(2)}{1} = 20$$

b) steady state error:

$$\text{given i/p } R(s) = \frac{3}{s} - \frac{2}{s^2} + \frac{1}{3s^3}$$

$$r(t) = \mathcal{L}^{-1}\{R(s)\}$$

$$= \mathcal{L}^{-1}\left\{\frac{3}{s} - \frac{2}{s^2} + \frac{1}{3s^3}\right\}$$

$$= \left(3 - 2t + \frac{1}{3}\left[\frac{t^2}{2}\right]\right)u(t)$$

$$\begin{aligned}\mathcal{L}^{-1}\left\{\frac{1}{s}\right\} &= 1 \\ \mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\} &= t \\ \mathcal{L}^{-1}\left\{\frac{1}{s^3}\right\} &= \frac{t^2}{2}\end{aligned}$$

$$r(t) = \cancel{3 - 2t + \frac{1}{3}\left[\frac{t^2}{2}\right]}$$

$$= \underbrace{3u(t)}_{\text{unit step}} - \underbrace{2tu(t)}_{\text{unit ramp}} + \underbrace{\frac{1}{3} \frac{t^2}{2} u(t)}_{\text{unit parabolic}}$$

$$e_{ss} = \frac{3}{1+K_p} - 2 \frac{1}{K_v} + \frac{1}{3} \frac{1}{K_a}$$

$$= \frac{3}{1+\infty} - 2 \cdot \frac{1}{\infty} + \frac{1}{3} \cdot \frac{1}{20}$$

$$= 0 - 0 + \frac{1}{60}$$

$$\boxed{e_{ss} = \frac{1}{60}}$$

2) The OLTF of servo system with unity f.b are given below. Find K_p, K_v, K_a and steady state error for unit step, ramp & parabolic

i) $G(s) = \frac{20(s+2)}{s(s+1)(s+3)}$

ii) $G(s) = \frac{10}{(s+2)(s+3)}$

iii) $G(s) = \frac{10}{s^2(s+1)(s+2)}$

Sol:

i) $G(s) = \frac{20(s+2)}{s(s+1)(s+3)}$

$$K_p = \lim_{s \rightarrow 0} G(s) = \infty$$

$$K_v = \lim_{s \rightarrow 0} sG(s) = 40/3$$

$$K_a = \lim_{s \rightarrow 0} s^2G(s) = 0$$

for step $e_{ss} = \frac{1}{1+K_p} = \frac{1}{1+\infty} = 0$

for ramp $e_{ss} = \frac{1}{K_v} = 3/40$

for parabolic $e_{ss} = \frac{1}{K_a} = 1/0 = \infty$

i)

$$G(s) = \frac{10}{(s+2)(s+1)}$$

$$K_p = 10/6$$

$$K_v = 0$$

$$K_a = 0$$

$$e_{ss} = \frac{1}{1+K_p} = \frac{1}{1+10/6} = \frac{3}{8}$$

$$e_{ss} = 1/K_v = \infty$$

$$e_{ss} = 1/K_a = \infty$$

ii)

$$G(s) = \frac{10}{s^2(s+1)(s+2)}$$

$$K_p = \infty$$

$$K_v = \infty$$

$$K_a = \frac{10}{2}$$

$$e_{ss} = \frac{1}{1+K_p} = 0$$

$$e_{ss} = \frac{1}{K_v} = 0$$

$$e_{ss} = \frac{1}{K_a} = \frac{2}{10}$$

Effect of P, PI, PD controllers.

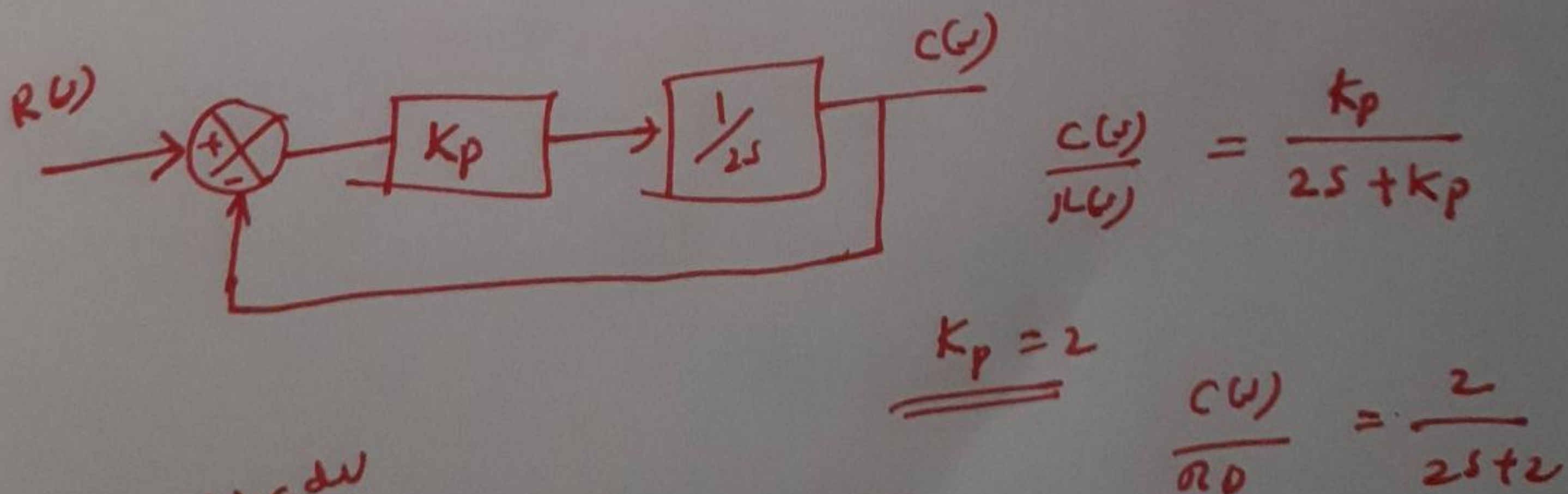
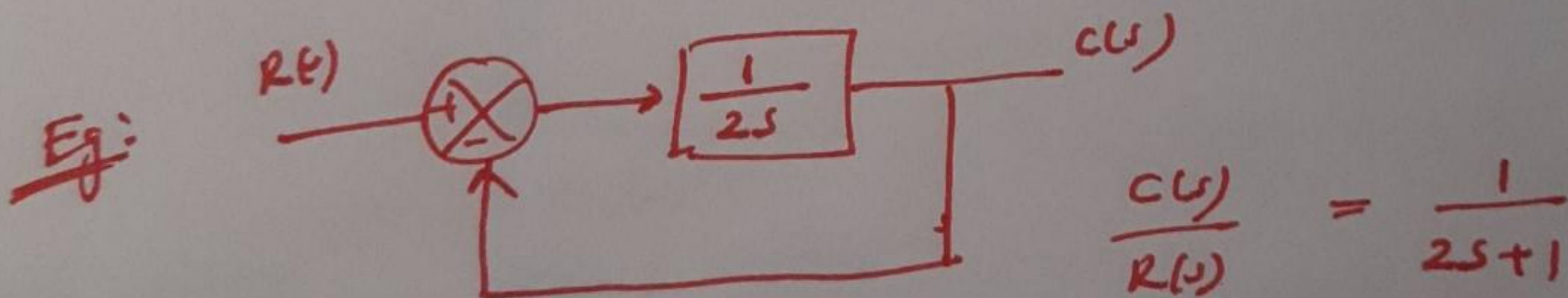
- To modify error signal
- To achieve better control action (stability)

Effect of proportional controller:-

produces an o/p signal which is proportional to error signal. The T.F of proportional controller is K_p , where K_p gain of controller

we are increasing the gain of the system

- i) Steady state accuracy ✓
- ii) Disturbance signal rejection ✓
- iii) Relative stability ✓



disadv

- i) decreases sensitivity
- ii) it produces constant s.s. error.

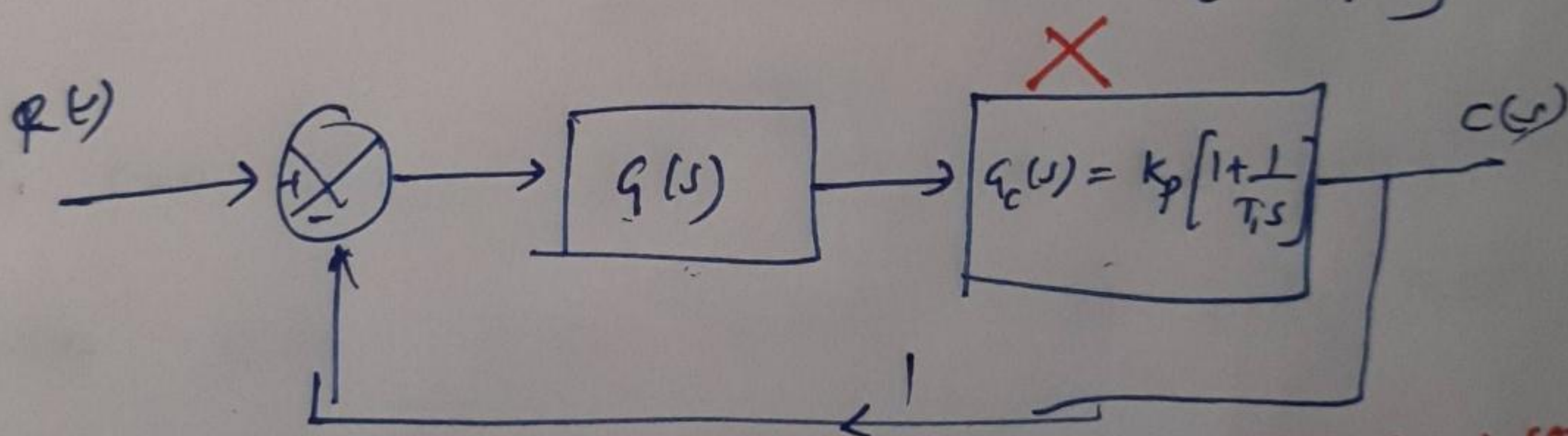
Effect of P-I controller:-

proportional plus Integral (PI) controller consists of two terms: one is proportional to error signal other is Integral of error signal.

$$\boxed{L.T \left\{ \int x(t) dt \right\} = \frac{X(s)}{s}}$$

T.F of P-I controller is $G_c(s) = \frac{K_p}{1} + \frac{K_p}{T_i s}$

$$= K_p \left[1 + \frac{1}{T_i s} \right]$$



No OLTF without controller

$$G(s) = \frac{\omega_n^2}{s(s + 2\xi\omega_n)}$$

without control

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\xi\omega_n s + \omega_n^2}$$

with control loop T.F

$$= G(s) \cdot G_c(s) \cdot H(s)$$

$$= \frac{\omega_n^2 \left[K_p \left(1 + \frac{1}{T_i s} \right) \right]}{s(s^2 + 2\xi\omega_n s)}$$

$$= \frac{K_p \omega_n^2 (1 + T_i s)}{s^2 T_i (s + 2\xi\omega_n)} \rightarrow (2)$$

$$\text{Now CLTF} = \frac{G(s)}{1 + G(s)H(s)}$$

$$= \frac{k_p \omega_n^2 (1 + T_i s)}{s^2 T_i (s + 2\xi \omega_n)}$$

$$1 + \frac{k_p \omega_n^2 (1 + T_i s)}{s^2 T_i (s + 2\xi \omega_n)}$$

CLTF with controller

$$\frac{C(s)}{R(s)} = \frac{k_p \omega_n^2 (1 + T_i s)}{s^3 + 2\xi \omega_n s^2 + k_p \omega_n^2 s + k_p \omega_n^2}$$

③

i) we can observe that order of the system will increase with P-I controller. If order increases then stability of system will decrease.

ii) Type ^{number} — increase

Then steady state error — decrease.

without controller

$$K_c = 0 \Rightarrow P_c \neq \infty$$

with

$$\frac{1}{s(s+2)} \left(2 + \frac{2}{s} \right)$$

from

$$G(s) = \frac{1}{s(s+2)}$$

$$G_c(s) = \left(2 + \frac{2}{s} \right)$$

s-s

parabola ^{3/2} 4p

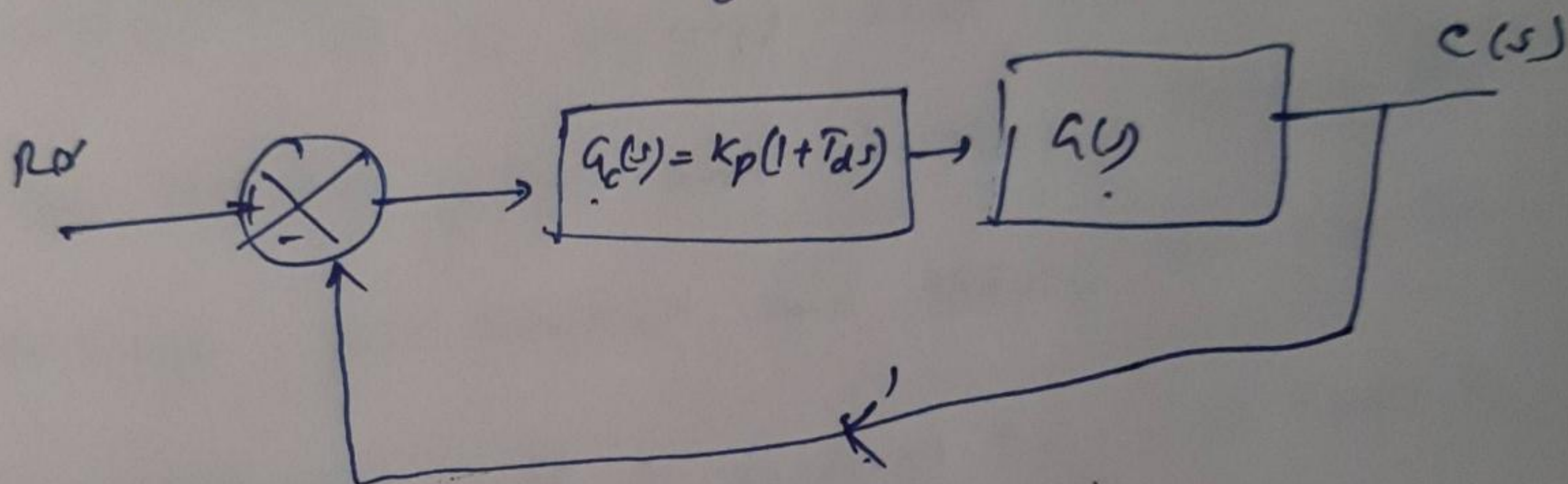
Effect of PD controller:-

It consists of 2 terms.

one is proportional to error signal.

other is proportional to derivative of error signal.

The T.F of P.D controller is $G_c(s) = K_p(1 + T_d s)$



Let OPLF of standard second system is

without controller OPLF $G(s) = \frac{\omega_n^2}{s(s + 2\xi\omega_n)}$ $\rightarrow$ ①

with controller loop T.F $= G(s)G_c(s) \cdot H(s)$
 $= \frac{\omega_n^2 [K_p(1 + T_d s)]}{s(s + 2\xi\omega_n)}$ $\rightarrow$ ②

with controller CLTF $\frac{C(s)}{R(s)} = \frac{\omega_n^2 [K_p(1 + T_d s)]}{s(s + 2\xi\omega_n)}$
 $1 + \frac{\omega_n^2 [K_p(1 + T_d s)]}{s(s + 2\xi\omega_n)}$

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2 (K_p + K_d s)}{s^2 + (2\xi\omega_n + K_p\omega_n^2 T_d)s + K_p\omega_n^2} \rightarrow \textcircled{3}$$

zero

where $K_d = K_p\omega_n^2 T_d$

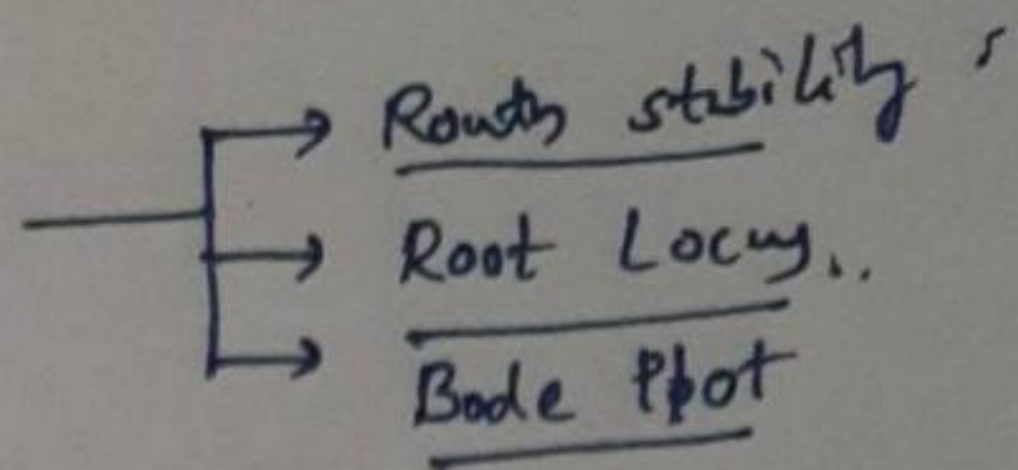
without controller

$$\text{CLTF} \quad \frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\xi\omega_n s + \omega_n^2} \rightarrow \textcircled{4}$$

From CLTF $\left[\textcircled{3} \& \textcircled{4} \right]$, it is observed that the PD controller introduces zero in the system and increases damping ratio. It will increase peak overshoot and reduces the rise time.

From OLTF $\left(\textcircled{1} \& \textcircled{4} \right)$, it is observed that PD controller does not modify the type number of the system. Hence PD controller will not modify steady state error.

Unit - III



Concepts of stability:

Def. for stability:

- i) A system is stable, if its output is bounded (finite) for any bounded input (finite)
- ii) A system is asymptotically stable, if in the absence of the i/p, the o/p tends to zero with zero initial conditions.
- iii) For a bounded i/p signal, if the output has constant amplitude oscillations then the system may be stable & undr. unstable. Such a system is called Limited stable.
- iv) If a system o/p is stable for all variations of its parameters then system is called absolutely stable.
- v) If the system is stable for a limited range of variations of its parameters, then that system is called Conditionally stable system.

Impulse Response:

Let $M(\omega)$ = CLTF of a system

$C(\omega)$ = output

$R(\omega)$ = input

$$M(\omega) = \frac{C(\omega)}{R(\omega)}$$

$$C(\omega) = M(\omega) \cdot R(\omega)$$

Now response in time domain $c(t) = \mathcal{L}^{-1}\{C(\omega)\}$

input in time domain $r(t) = \mathcal{L}^{-1}\{R(\omega)\}$

For an impulse input $r(t) = \delta(t) \Rightarrow R(\omega) = \mathcal{L}\{\delta(t)\} = 1$

$$\text{Impulse response} = \mathcal{L}^{-1}\{C(\omega)\}$$

$$= \mathcal{L}^{-1}\{M(\omega) \cdot R(\omega)\}$$

$$= \mathcal{L}^{-1}\{M(\omega)\}$$

$$= m(t)$$

Impulse response of the system is inverse

L-T of system T-F

Location of poles on s-plane for stability

Eg:

$$\frac{C(s)}{R(s)} = \frac{G(s)}{1 + G(s)H(s)}$$

To find poles $\Rightarrow$ equate denominator to zero
(characteristic equation)

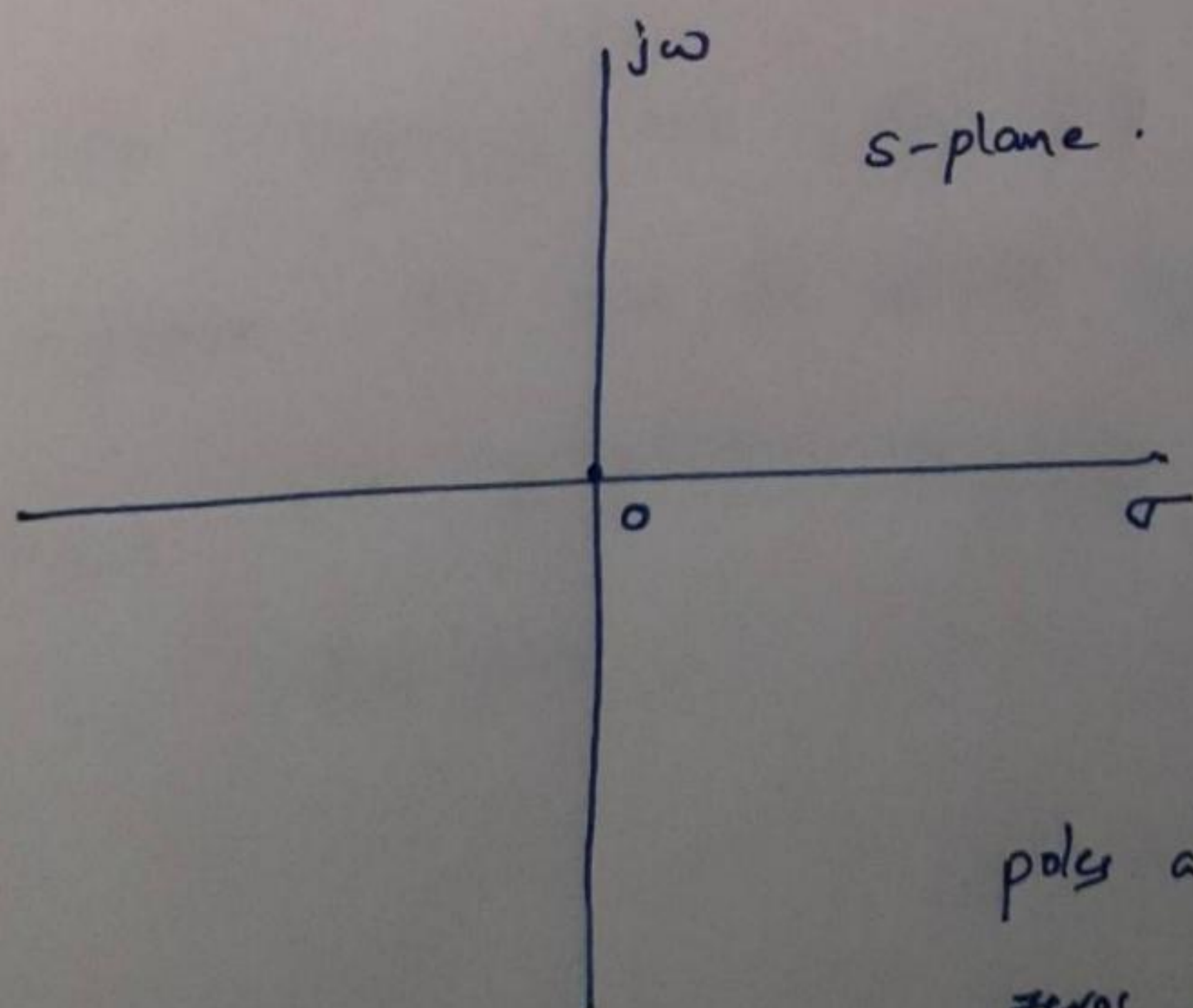
$$\text{denominator of CLTF} = \text{characteristic equation} \\ = 1 + G(s)H(s)$$

To find zeros $\Rightarrow$ equate numerator to zero

s-plane:

x-axis $\rightarrow$ Real axis (σ)

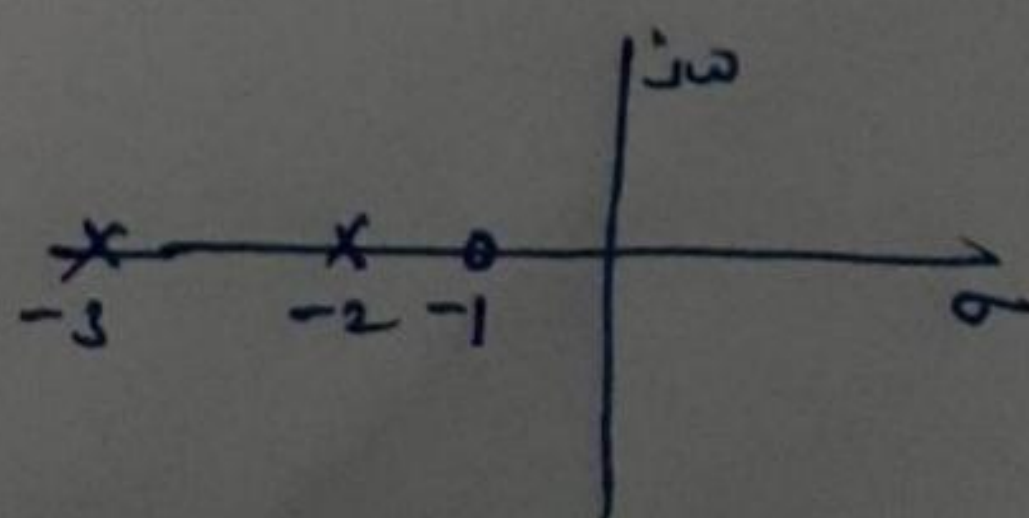
y-axis $\rightarrow$ Imag axis ($j\omega$)



poles are indicated by 'X'
zeros " " " 'O'

$$\frac{C(s)}{R(s)} = \frac{s+1}{(s+2)(s+3)}$$

poles $s = -2, -3$
zeros $s = -1$



The CLTF $M(s)$ can be expressed as ratio of two polynomials in 's'. The denominator polynomial of CLTF is called characteristic equation (C.E) of the system.

The roots of C.E are poles of CLTF.

$$M(s) = \frac{b_0 s^m + b_1 s^{m-1} + b_2 s^{m-2} + \dots + b_m}{a_0 s^n + a_1 s^{n-1} + a_2 s^{n-2} + \dots + a_n}$$

$$= \frac{(s+z_1)(s+z_2) \dots (s+z_n)}{(s+p_1)(s+p_2) \dots (s+p_n)}$$

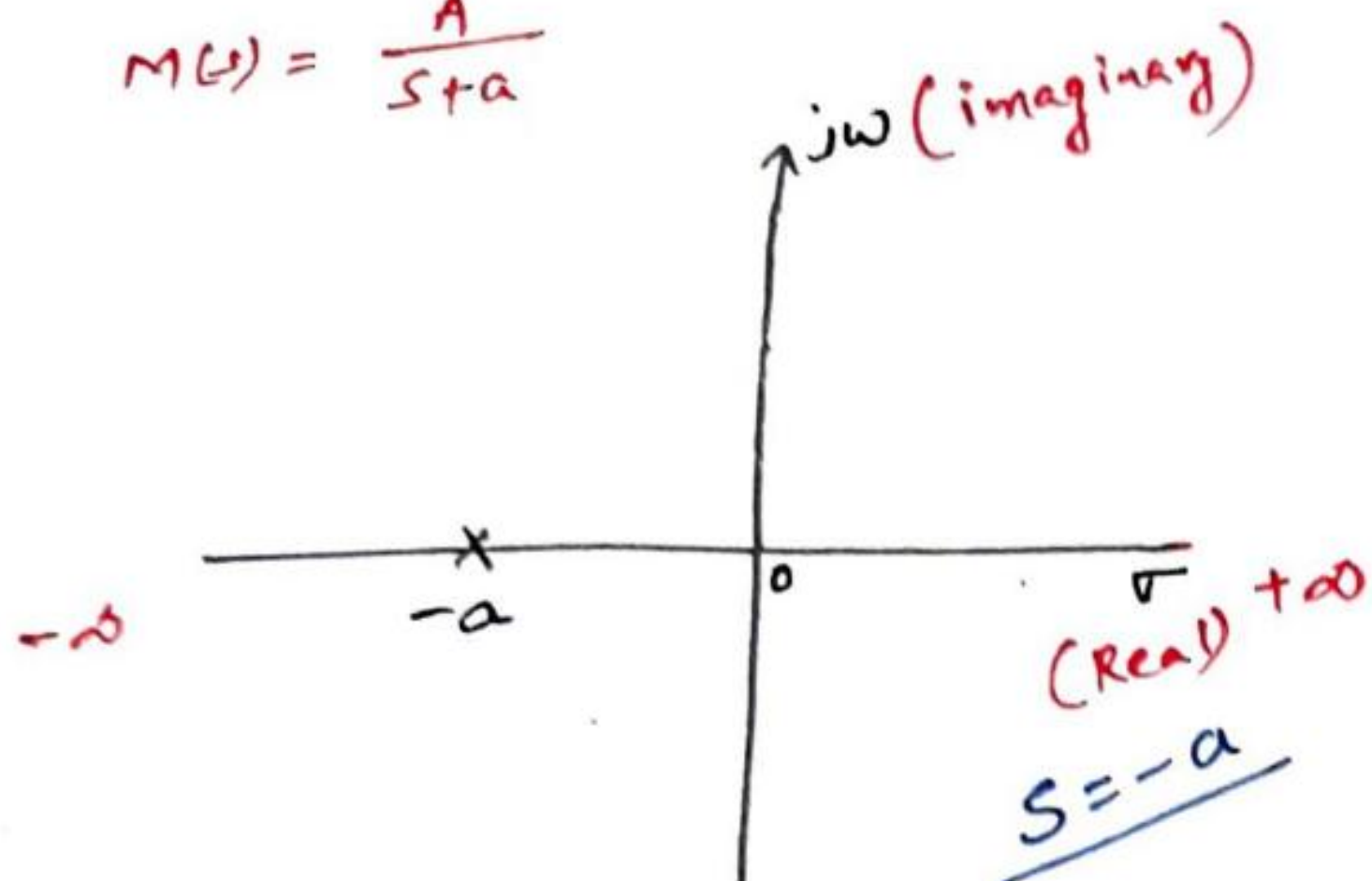
The roots of numerator polynomial are zeros ($z_1, z_2, \dots$)

The roots (or poles) $p_1, p_2, \dots, p_n$ may be at origin (or) lying on imaginary axis (or) lying on Left/Right half of s-plane. It can be found by using impulse response.

The inverse LT of $M(s)$ depends on location of poles in s-plane.

Transfer function $M(s)$ and
location of roots on s -plane

i) $M(s) = \frac{A}{s+a}$

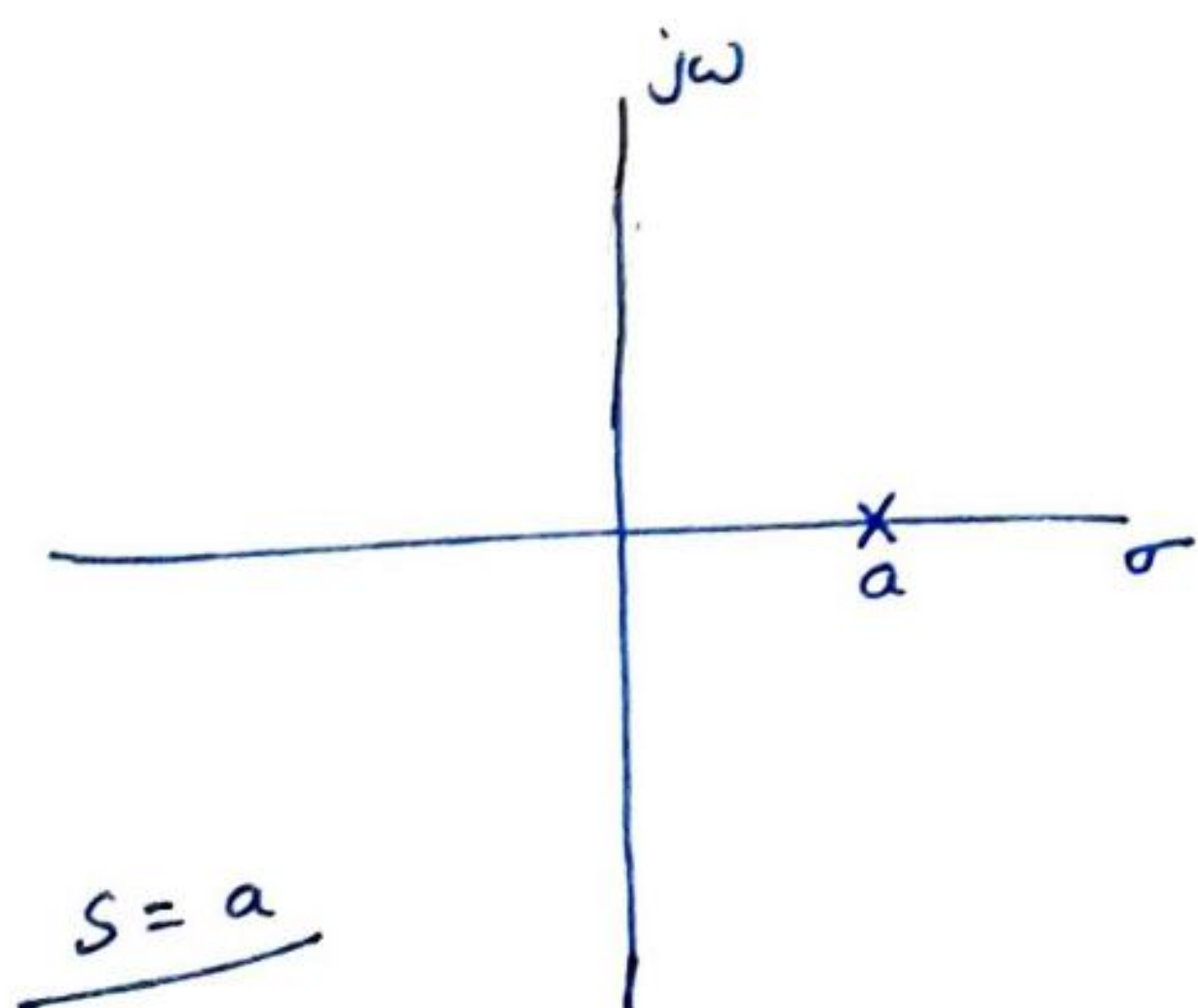


Root on negative Real axis

C.E $\Rightarrow s+a=0$
 $s=-a$

$\rightarrow$ If all the roots of C.E have negative real parts (i.e. poles are lying on Left Half of s -plane) then its impulse response is bounded (decreases to zero when $t \rightarrow \infty$) then system is stable.

$M(s) = \frac{A}{s-a}$



Root on positive real axis.

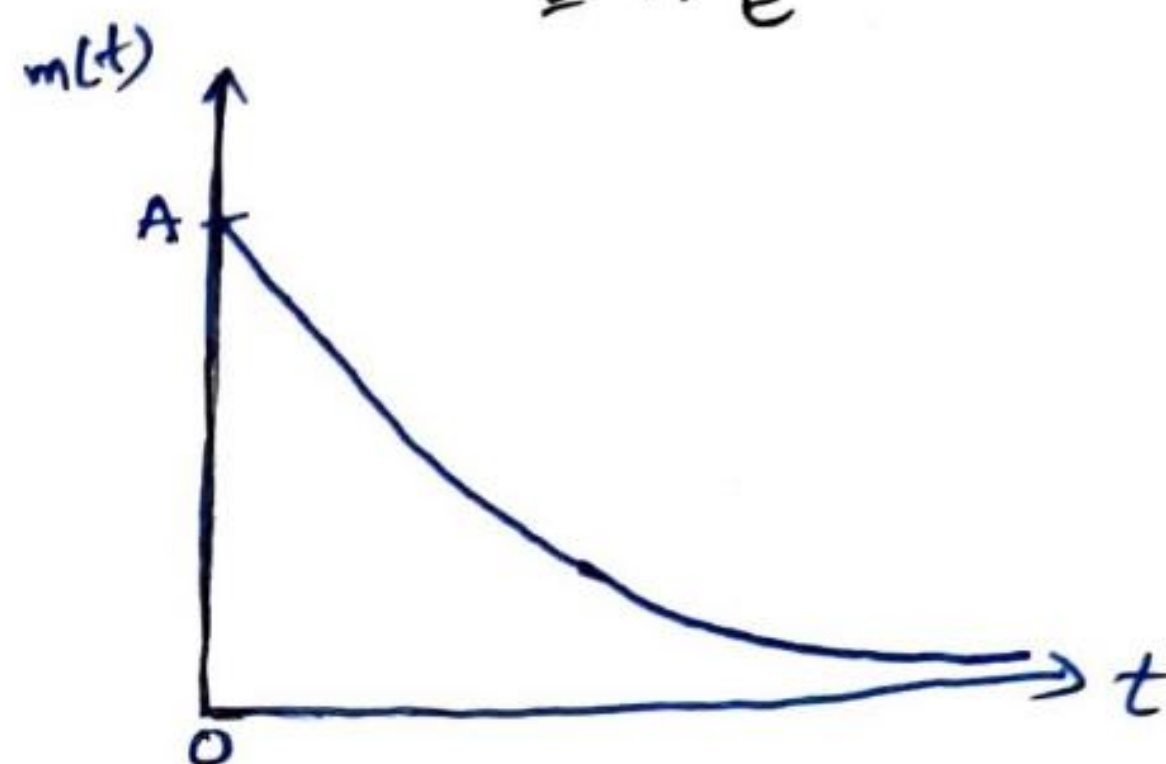
$\rightarrow$ If any root of C.E has a positive real part (i.e. poles are lying on Right Half of s -plane) then impulse response unbounded (unstable system)

Impulse Response

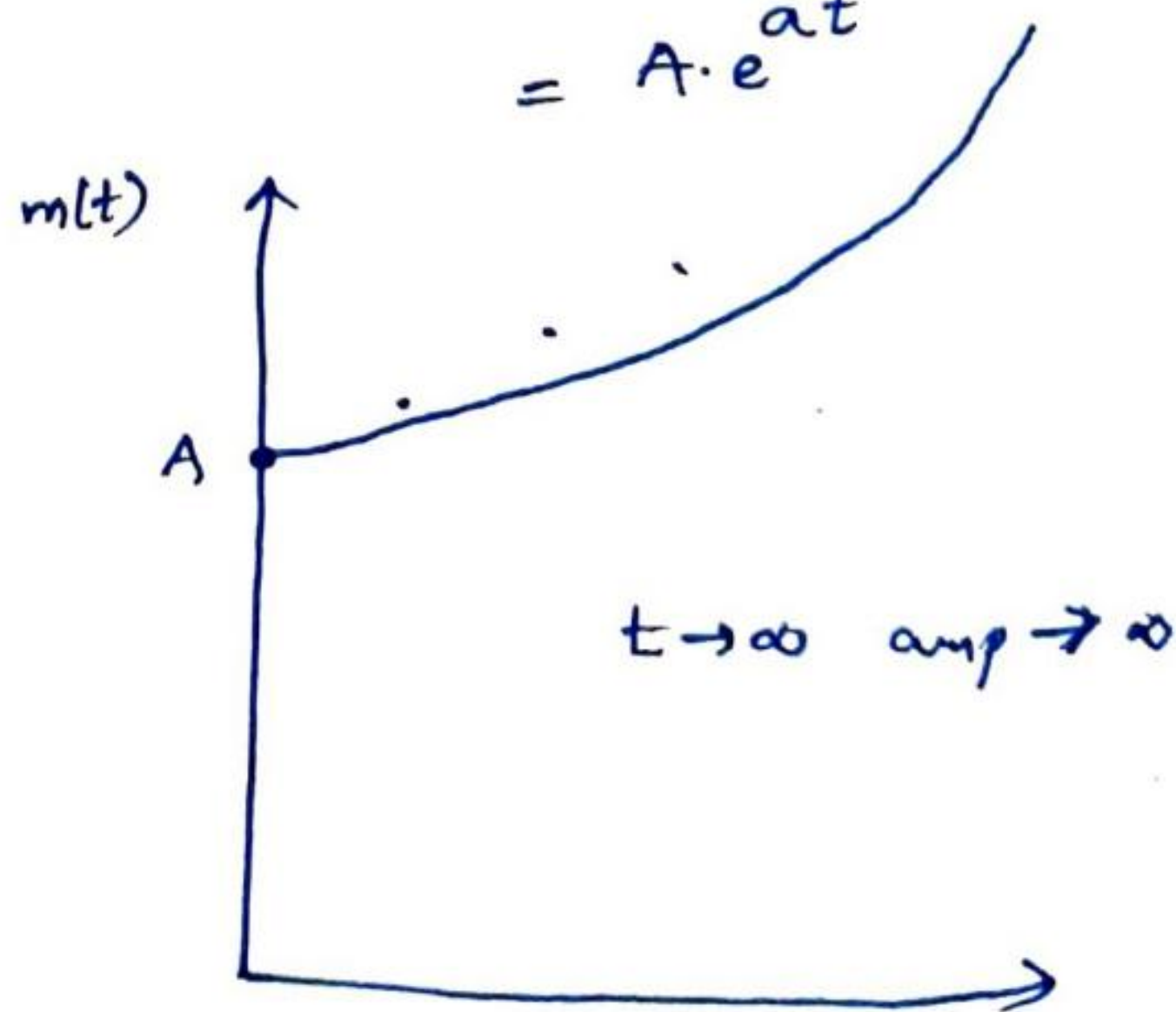
$m(t)$

$t \rightarrow \infty \Rightarrow \text{Amp} = \text{finite} \Rightarrow \text{stable}$
 $t \rightarrow \infty \Rightarrow = \infty \Rightarrow \text{unstable}$
if amp is constant for infinite amount of time it is marginally stable
 $e^{-\infty} = 0$ $e^{\infty} = \infty$

$\mathcal{L}^{-1}\{M(s)\} = m(t) = \mathcal{L}^{-1}\left\{\frac{A}{s+a}\right\}$
 $= A \cdot e^{-at}$

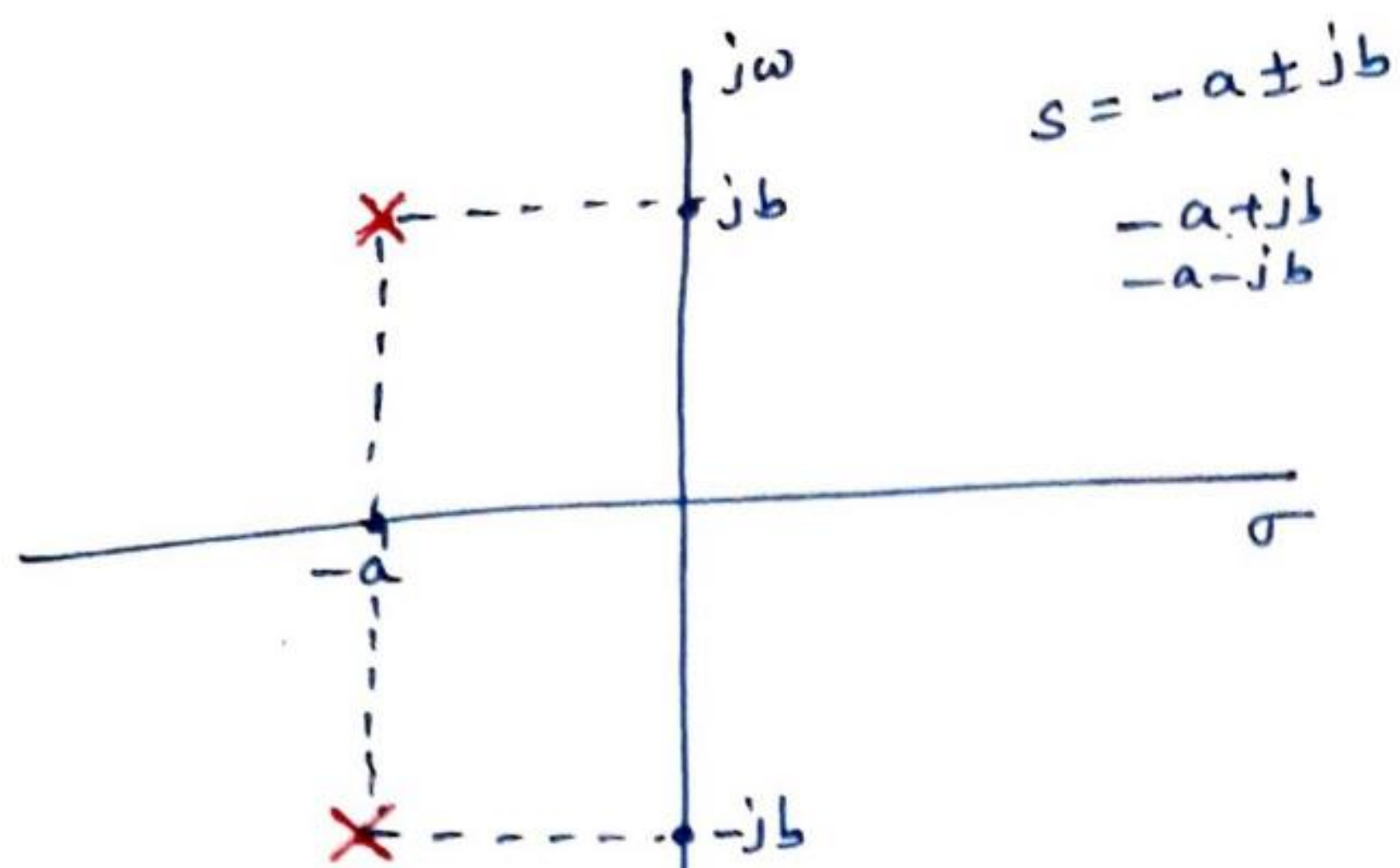


$m(t) = \mathcal{L}^{-1}\{M(s)\} = \mathcal{L}^{-1}\left\{\frac{A}{s-a}\right\}$
 $= A \cdot e^{at}$



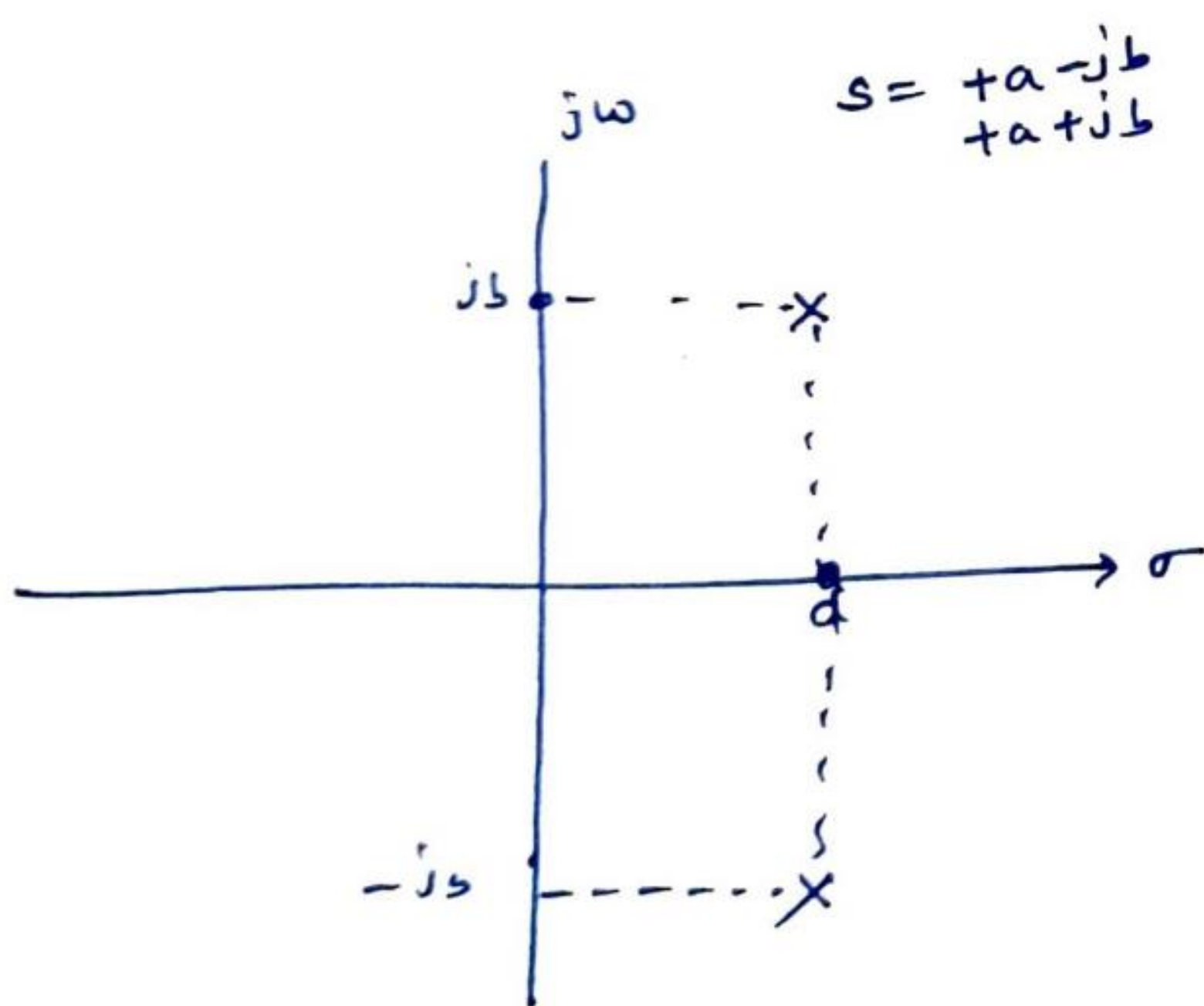
unstable system

iii) $M(s) = \frac{A}{s+a+jb} + \frac{A^*}{s+a-jb}$



complex roots on left half of s-plane
conjugate

iv) $M(s) = \frac{A}{s-a+jb} + \frac{A^*}{s-a-jb}$

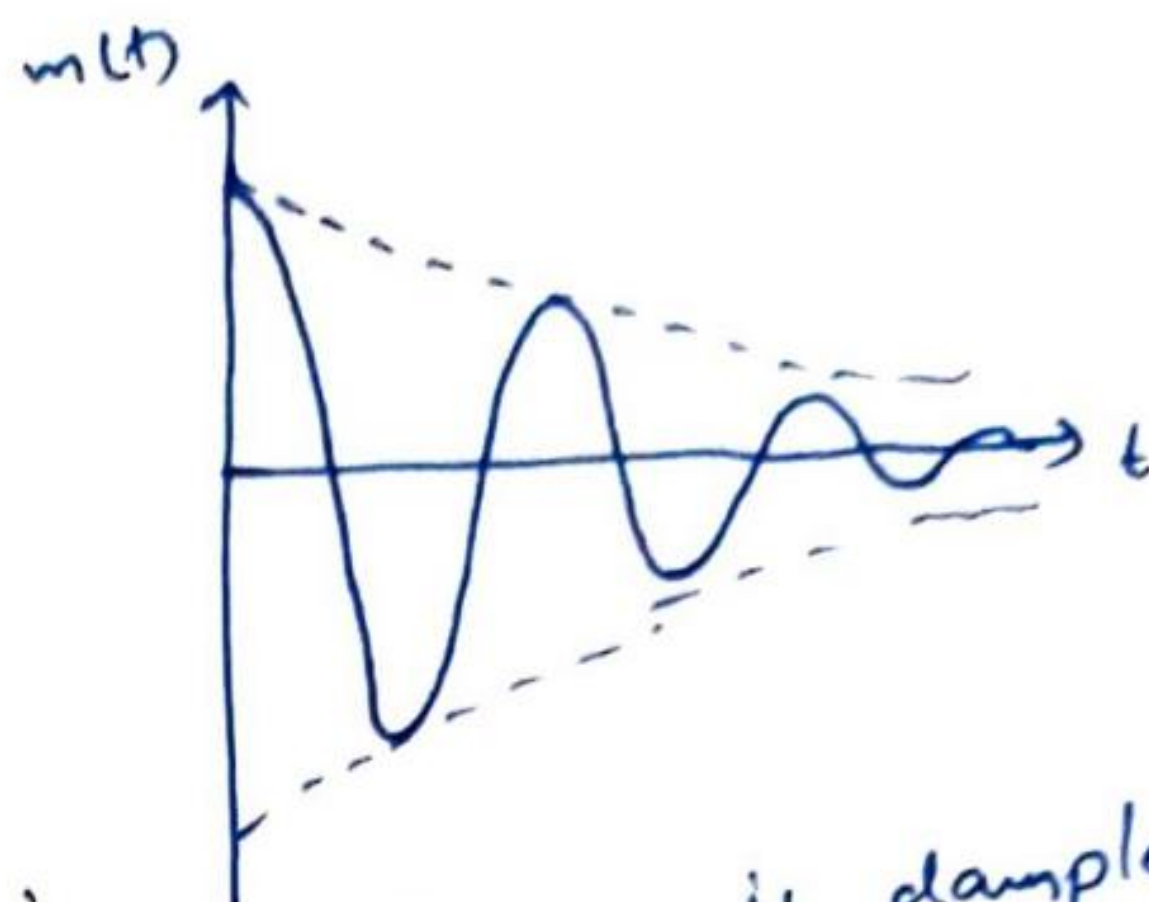


complex conjugate roots lying on
Right half of s-plane.

$$m(t) = \mathcal{L}^{-1}\{M(s)\}$$

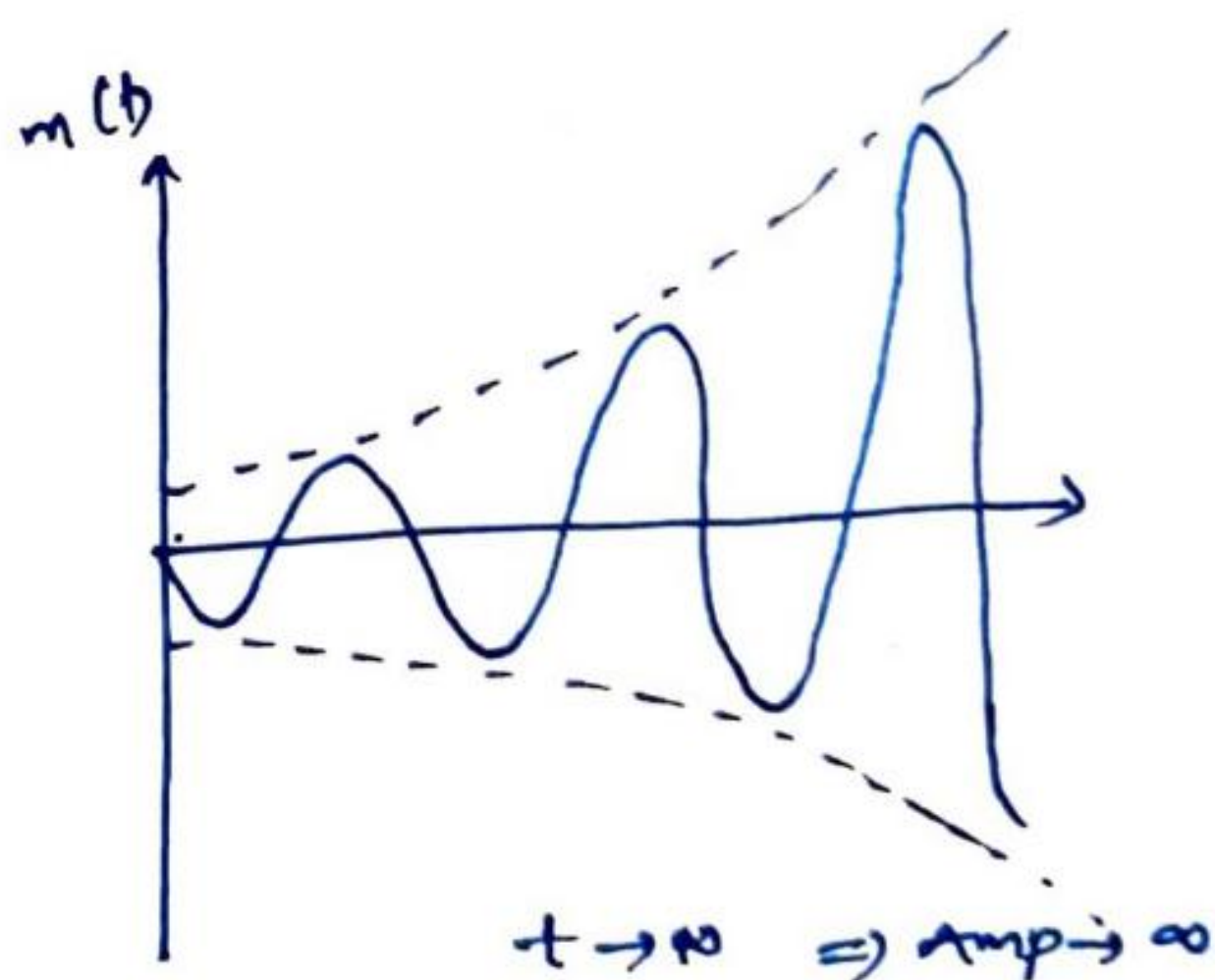
$$= \mathcal{L}^{-1}\left\{\frac{A_1}{(s+a)^2 + b^2}\right\}$$

$$= A \cdot e^{-at} \cos bt$$



impulse response is damped
sinusoidal. system is stable

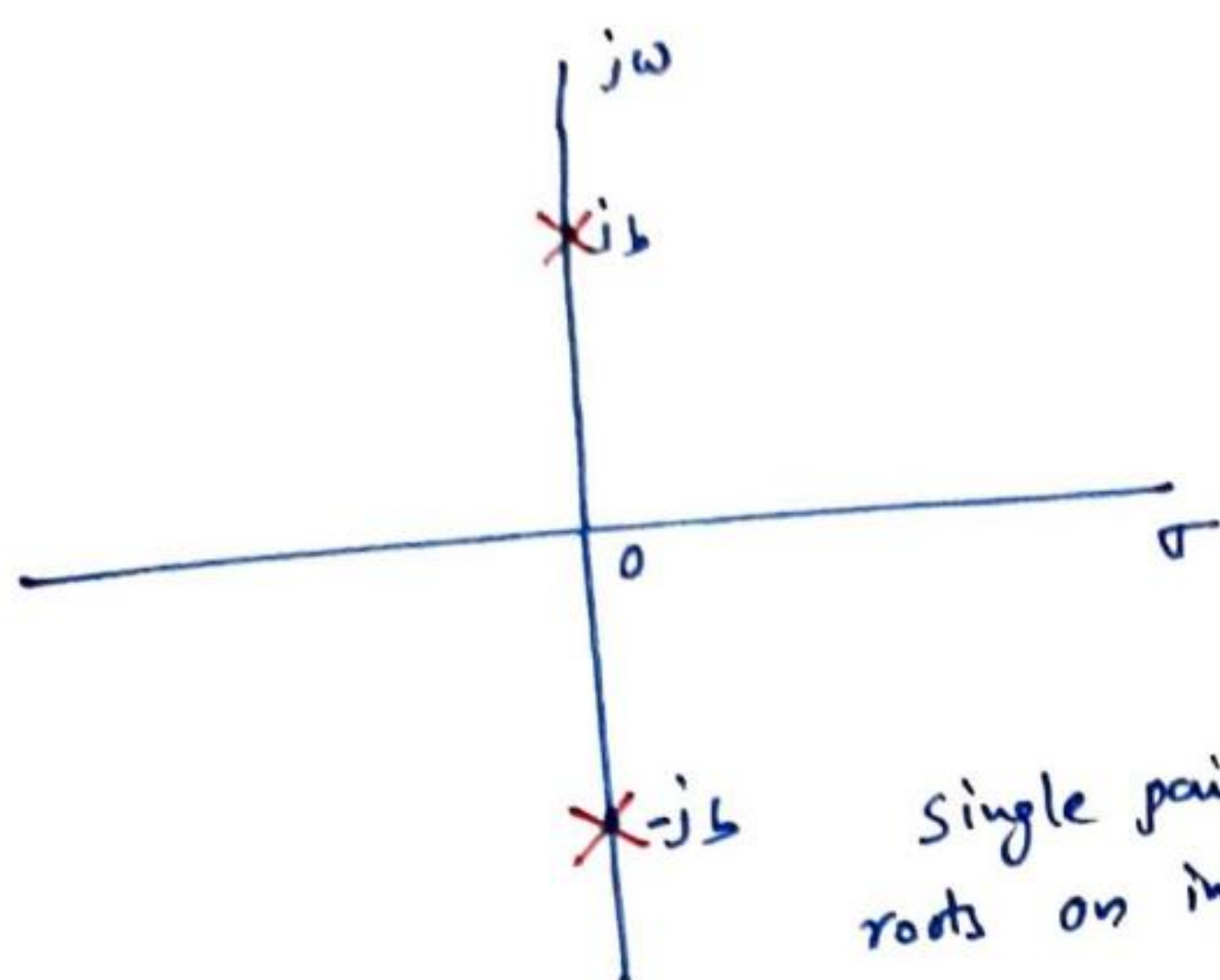
$$m(t) = \mathcal{L}^{-1}\{M(s)\} = A \cdot e^{at} \cos bt$$



impulse response is exponentially
increasing sinusoidal response.
system is unstable.

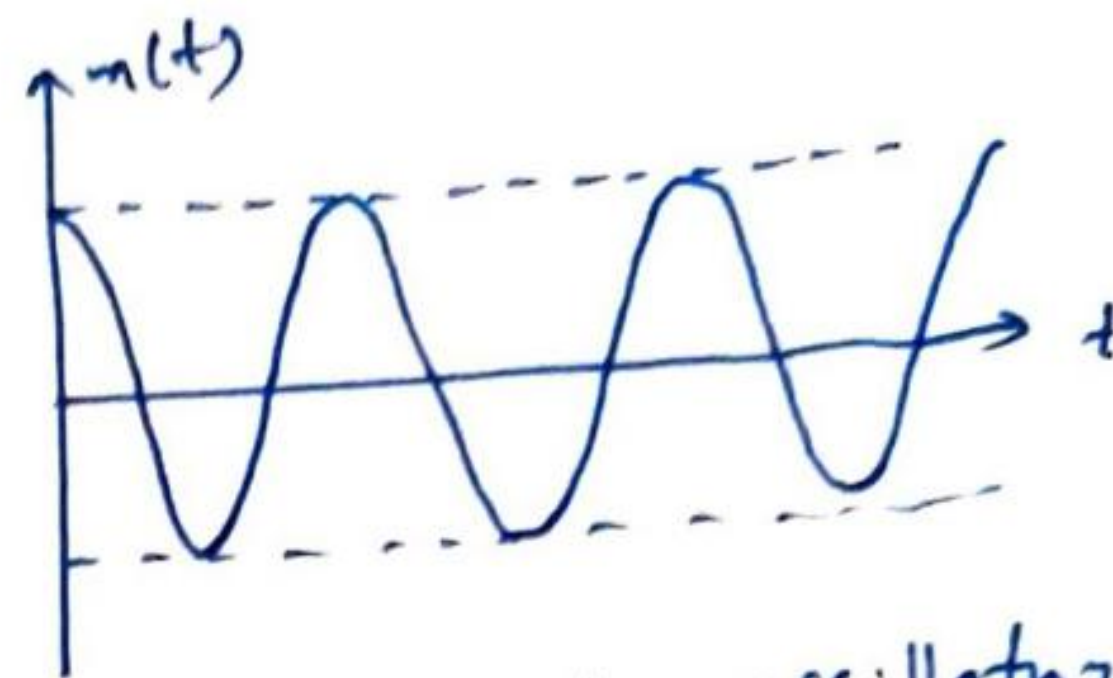
$$v) \quad M(s) = \frac{A}{s+jb} + \frac{A^*}{s-jb}$$

$$s = \begin{matrix} -jb \\ +jb \end{matrix}$$



single pair of roots on img axis

$$m(t) = \mathcal{L}^{-1}\{M(s)\} = \mathcal{L}^{-1}\left\{\frac{A}{s^2+b^2}\right\} = A \cdot \cos bt$$

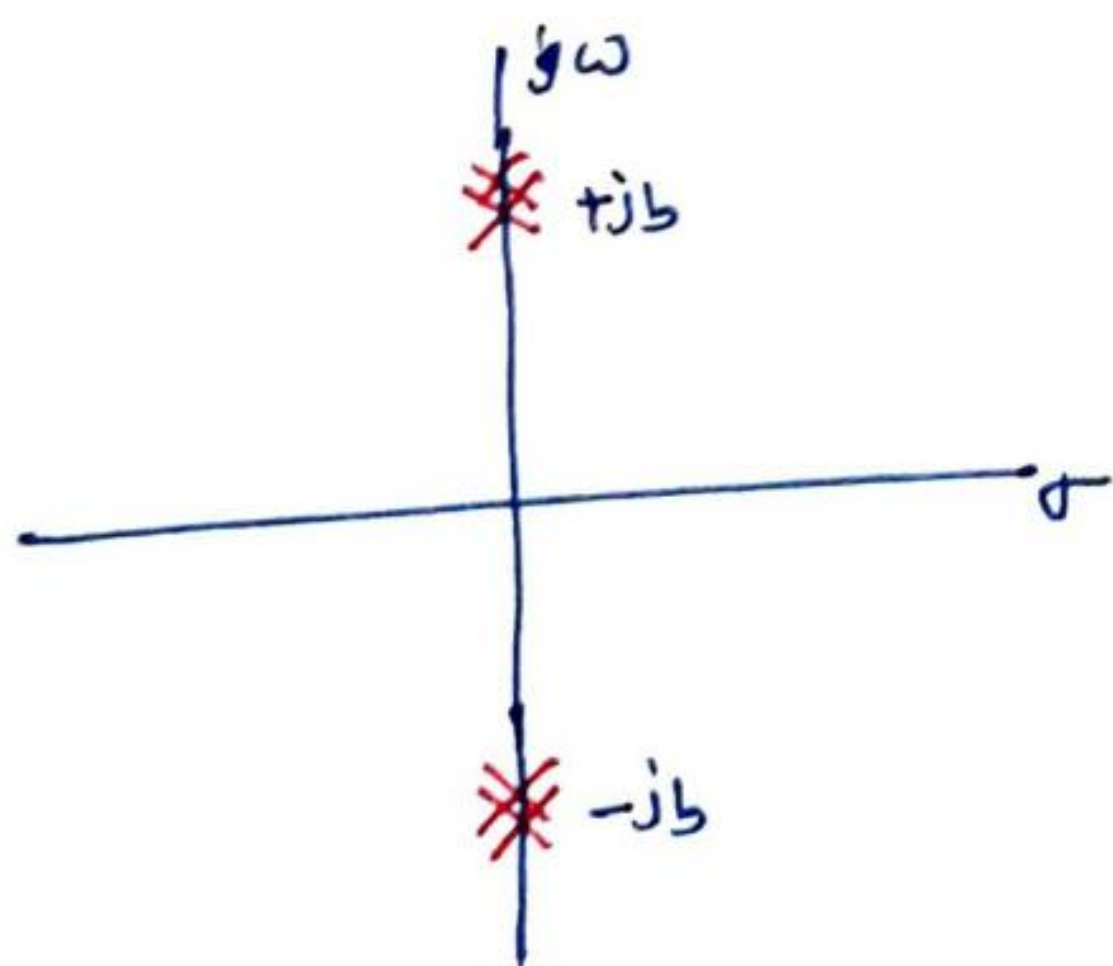


Impulse Response is oscillating system is marginally stable system

→ If one or more non-repeated roots of C.E are lying on imaginary axis, then the impulse response is bounded. system is marginally stable.

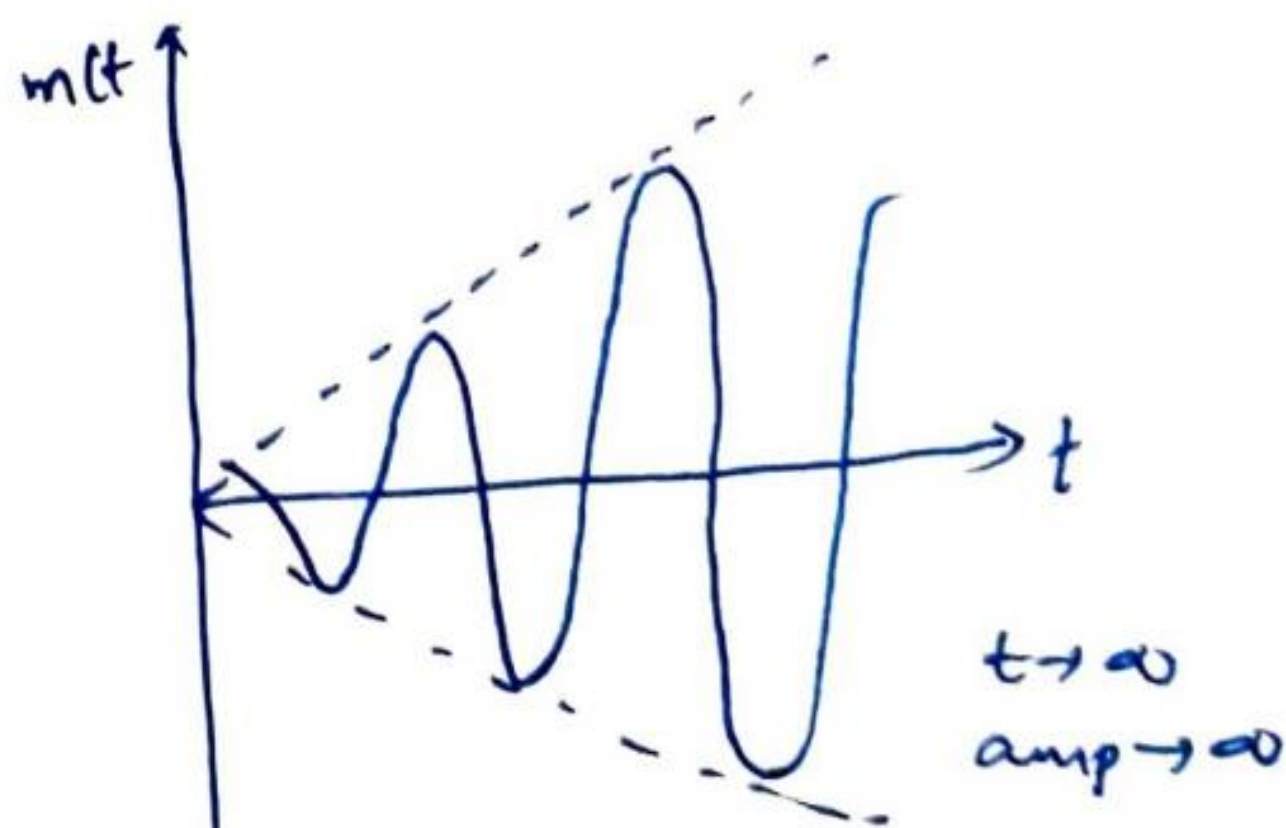
$$vi) \quad M(s) = \frac{A}{(s+jb)^2} + \frac{A^*}{(s-jb)^2}$$

$$s = \begin{matrix} +jb \text{ (2 roots)} \\ -jb \text{ (2 roots)} \end{matrix}$$



Double pair of roots on img. axis.

$$m(t) = \mathcal{L}^{-1}\left\{\frac{A}{(s+jb)^2} + \frac{A^*}{(s-jb)^2}\right\} = A t e^{-jbt} + A^* t e^{jbt} = 2At \cdot \cos bt$$

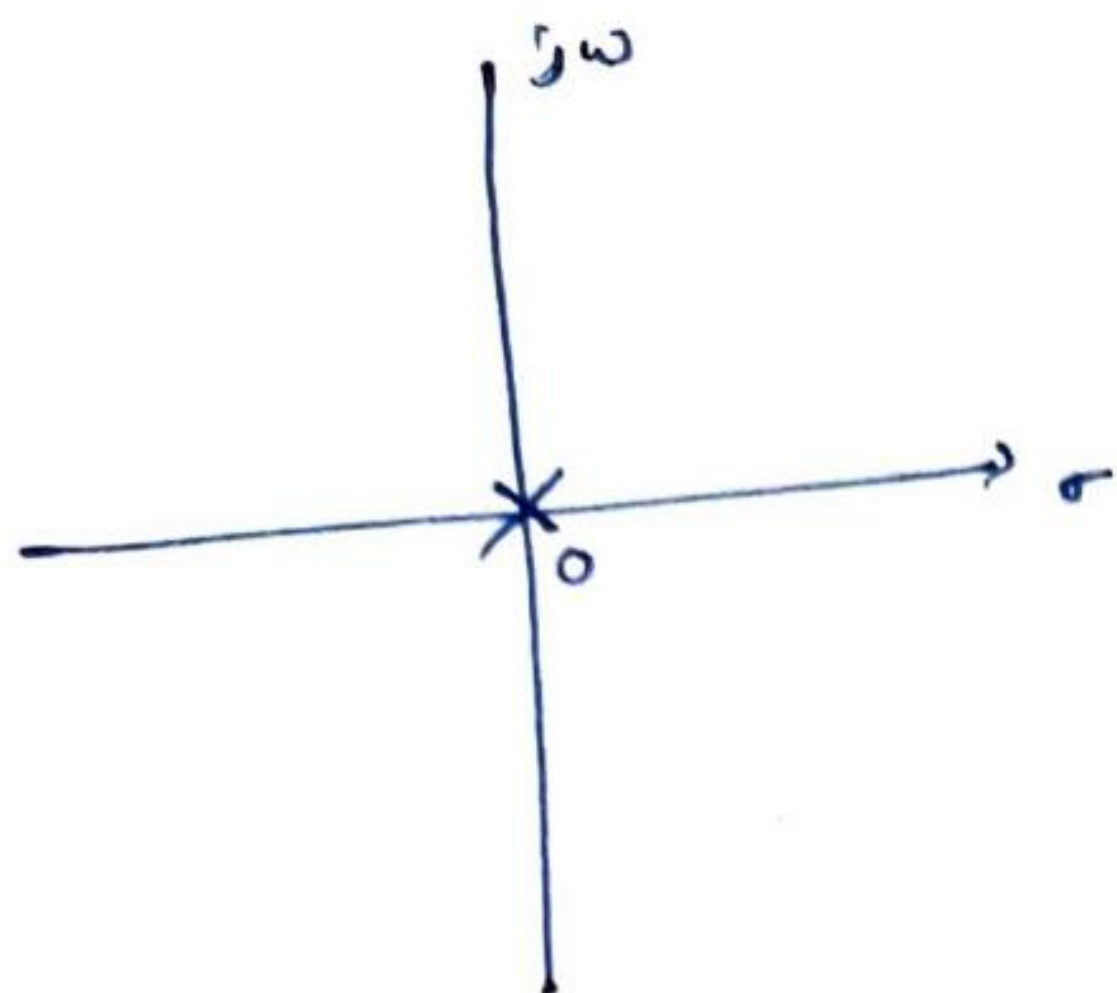


impulse response is linearly increasing sinusoidal response. unstable

→ If the C.E has repeated roots on img. axis then impulse response is unbounded. system is unstable system.

$$M(s) = \frac{A}{s}$$

$$s = 0$$

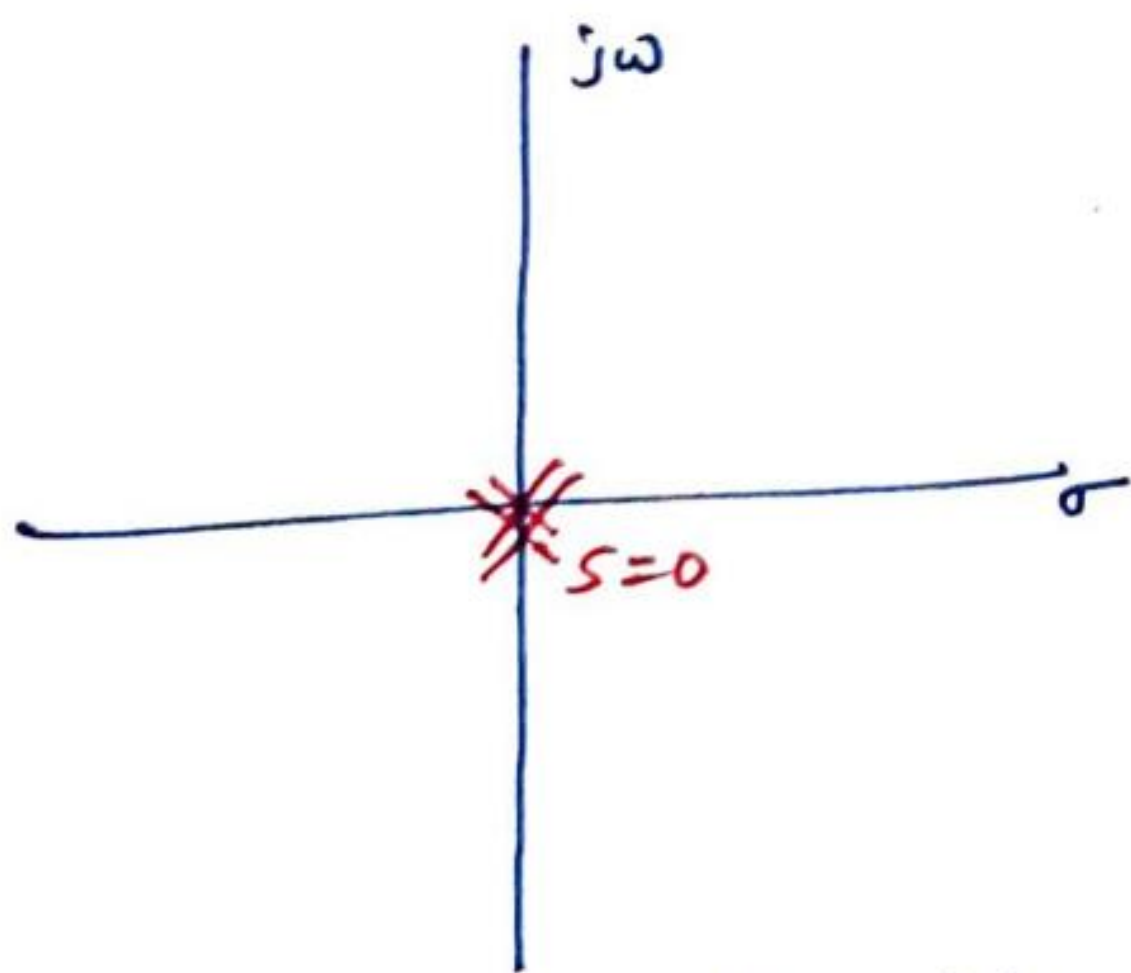


single root at origin

→ If the C.E has single root at origin, then impulse response is bounded. system is marginally stable system.

$$M(s) = \frac{A}{s^2}$$

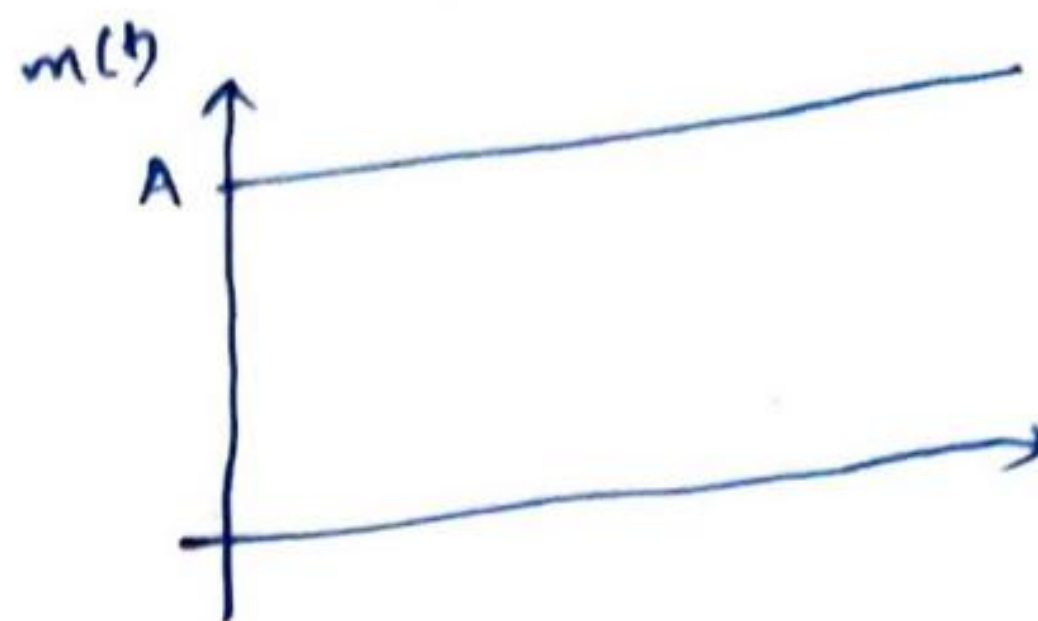
$$s = 0, 0$$



Double root at origin.

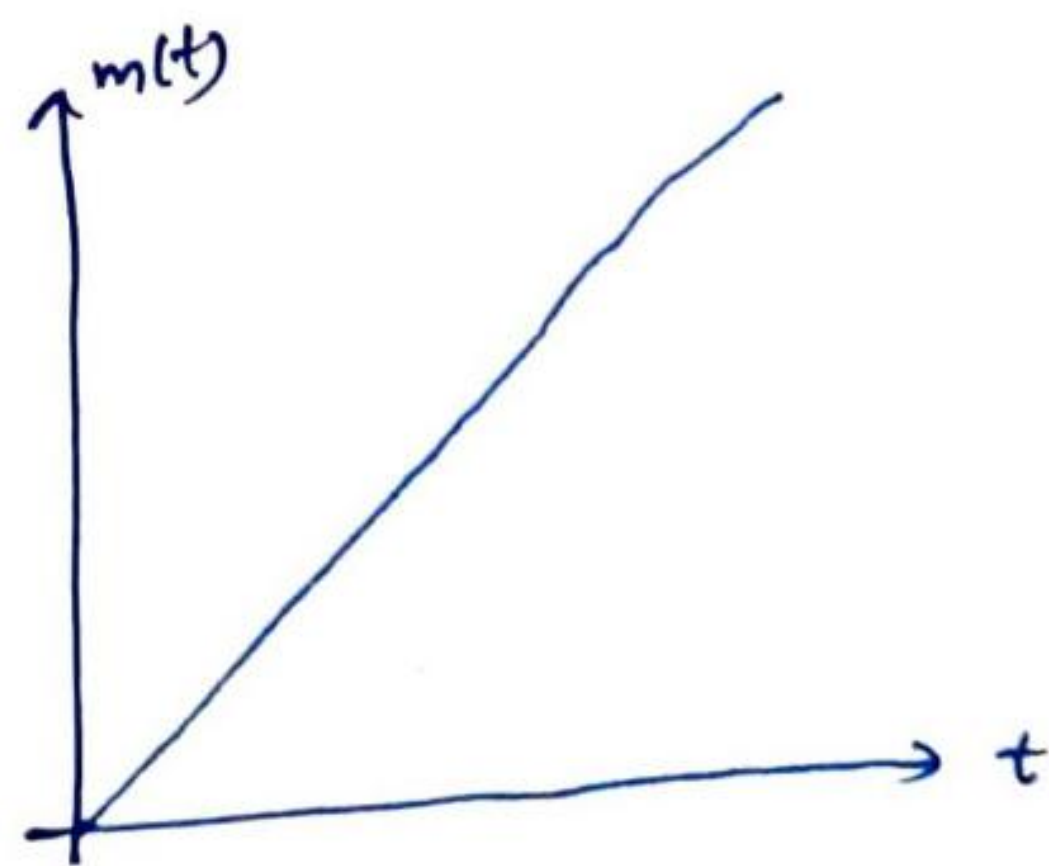
→ If the C.E has repeated roots at origin then impulse response is unbounded. system is unstable system.

$$m(t) = \mathcal{L}^{-1}\{M(s)\} = \mathcal{L}^{-1}\left\{\frac{A}{s}\right\} = A u(t)$$



impulse response is constant
system is marginally stable system.

$$m(t) = \mathcal{L}^{-1}\{M(s)\} = \mathcal{L}^{-1}\left\{\frac{A}{s^2}\right\} = At$$

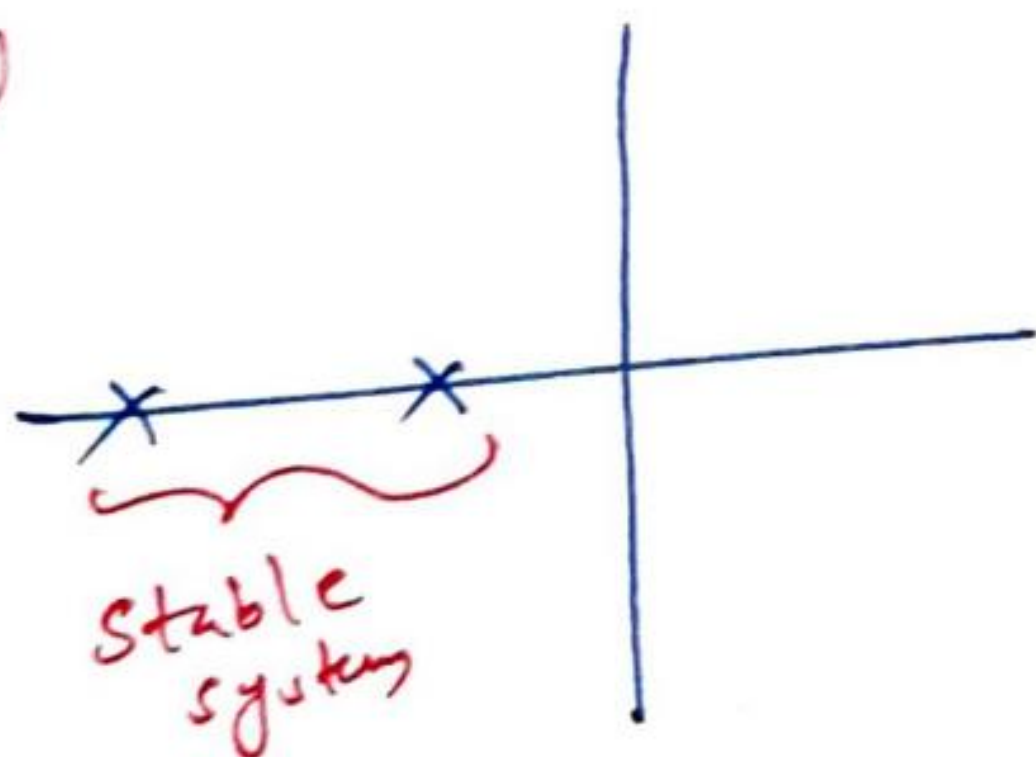


impulse response linearly increases
system is unstable

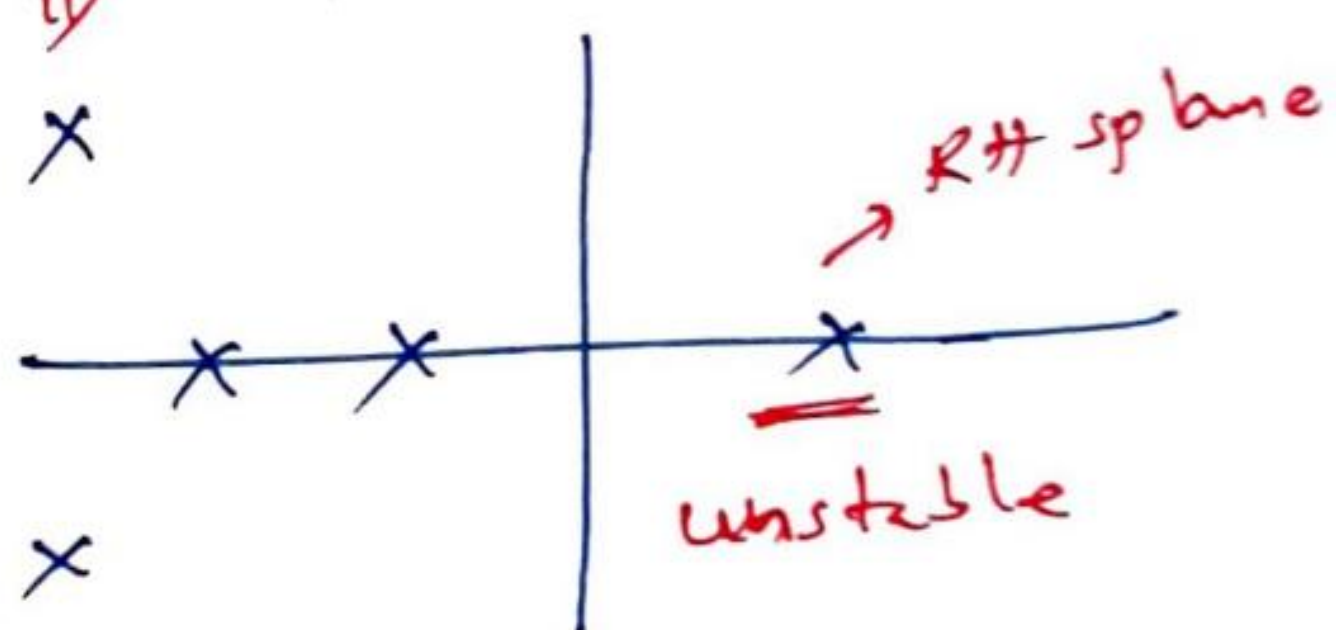
Note: stability, depending on location of roots of C.E

- 1) If all the roots of C.E has negative real parts, then the system is stable system.
- 2) If any root of C.E has positive real part (or) if there is a repeated root on imaginary axis then the system is unstable.
- 3) If the condition (1) is satisfied except for the presence of one or more non repeated roots on imaginary axis, then the system is limitedly (or) marginally stable.

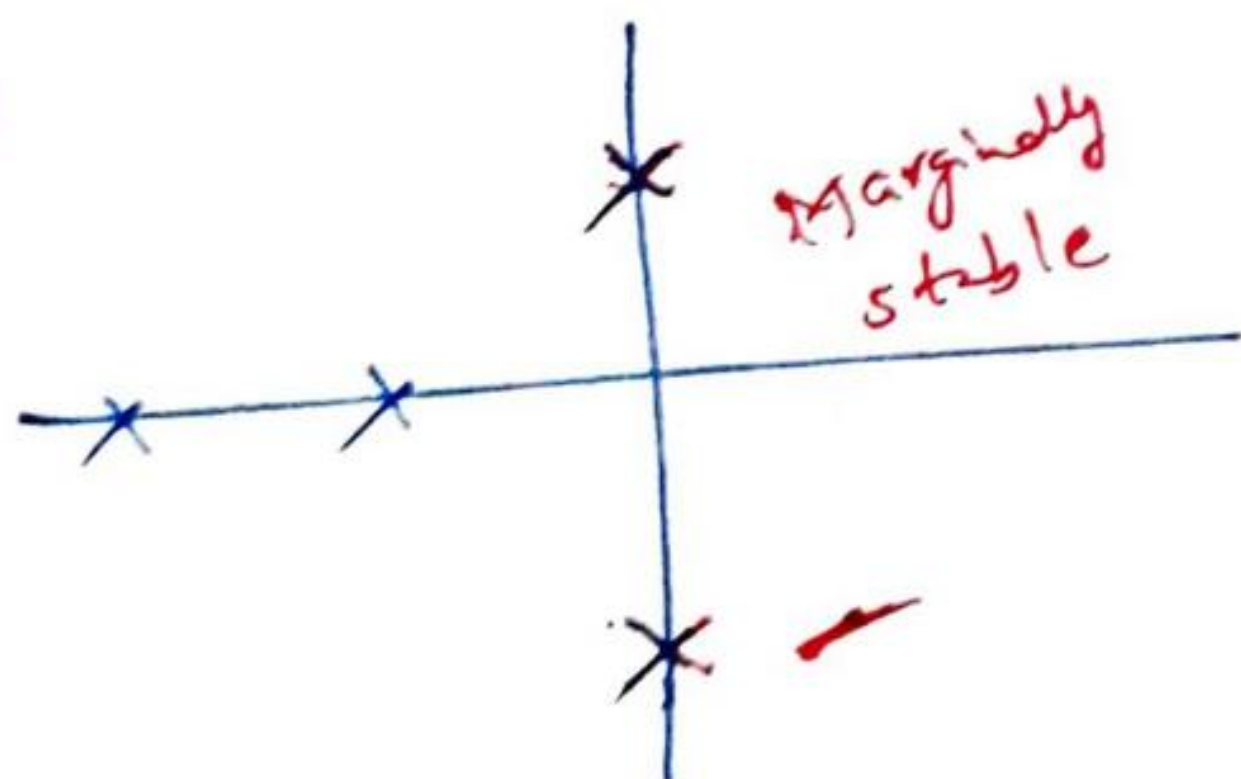
Eg: i)



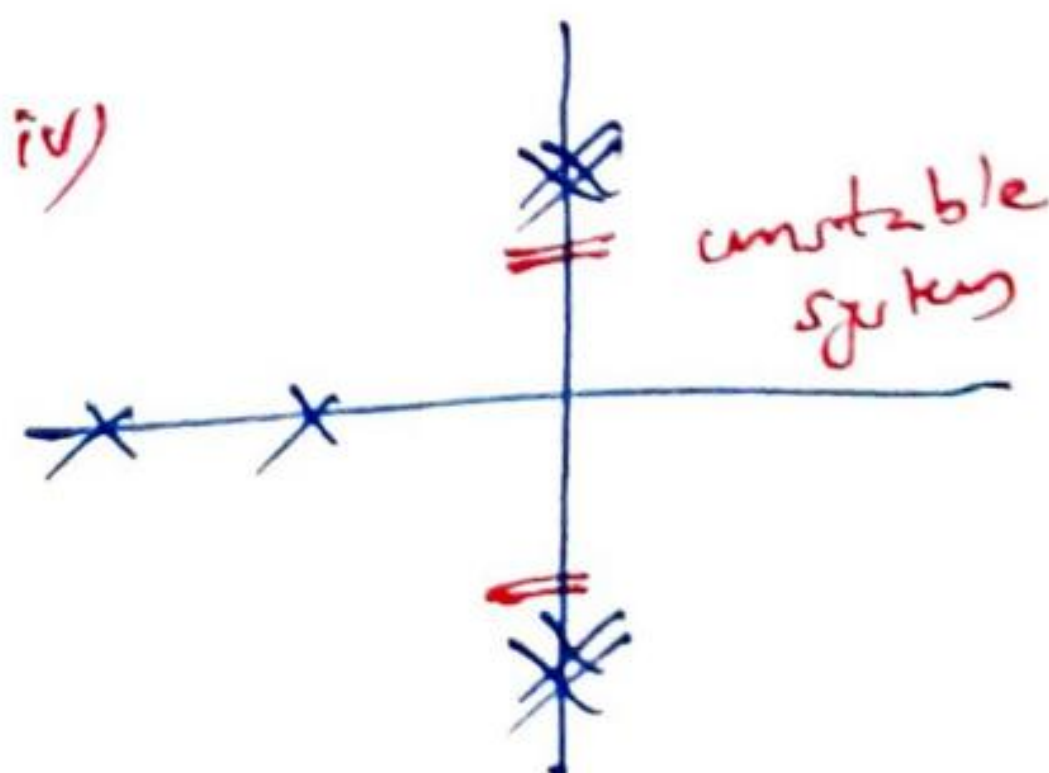
ii)



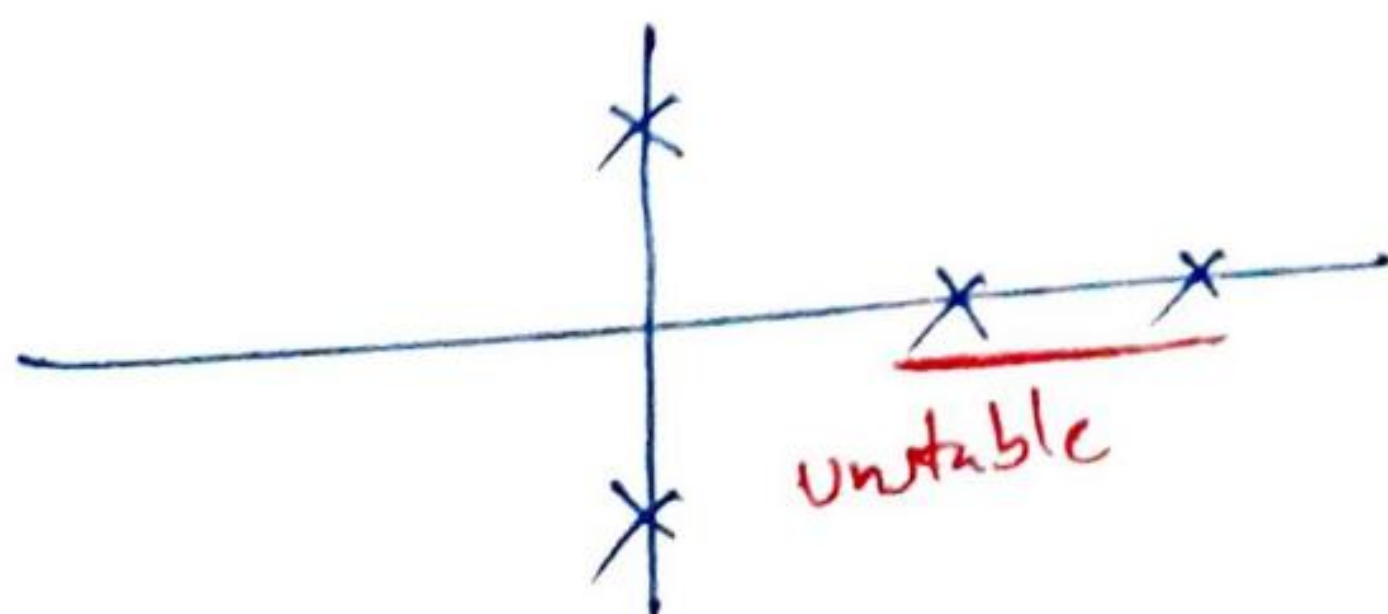
iii)



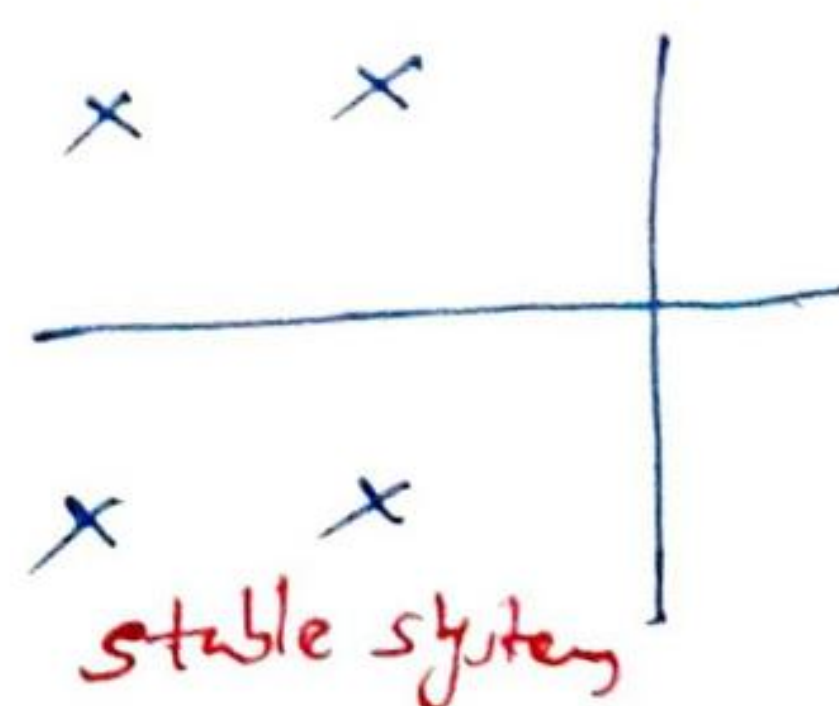
iv)



v)



vi)



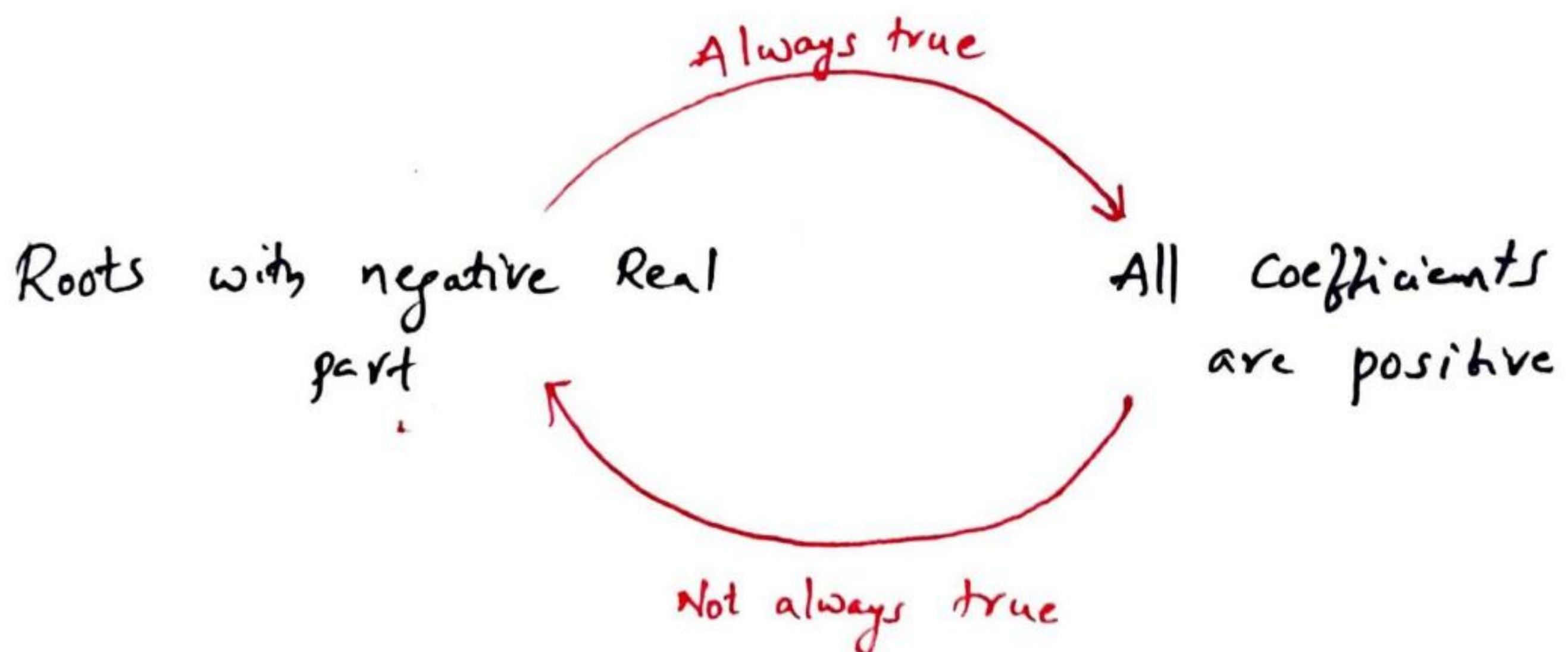
Consider the n^{th} order C.E ,

$$a_0 s^n + a_1 s^{n-1} + a_2 s^{n-2} + \dots + a_{n-1} s + a_n = 0$$

$$\text{C.E} \Rightarrow 1 + G(s) \cdot H(s) = 0$$

By using coefficients of C.E , we can make the following conclusions .

- i) If all the coefficients are positive and if no coefficient is zero, then all roots are in the Left half of s-plane.
- ii) If any coefficient is equal to zero then some of the roots may be on the imag axis. (or) R.H of s-plane
- iii) If any coefficient is negative, then at least one root is in the Right Half of s-plane.



Routh Hurwitz Criterion :-

It is an analytical procedure for determining whether all the roots of C.E have negative real part or not.

A. Hurwitz and E.J Routh independently published the method of investigating the sufficient condition of a stability of a system.

The Routh stability criterion can be stated as follows

"The necessary and sufficient condition for stability is that all of the elements in the first column of Routh array be positive. If this condition is not met, the system is unstable and the number of sign changes in the elements of first column of Routh array corresponds to the number of roots of the C.E in the Right half of s-plane."

The Routh stability criterion is based on ordering the coefficients of C.E into a schedule called Routh array shown below.

$$a_0 s^n + a_1 s^{n-1} + a_2 s^{n-2} + \dots + a_{n-1} s^1 + a_n = 0$$

s^n	a_0	a_2	a_4	...
s^{n-1}	a_1	a_3	a_5	...
s^{n-2}	b_0	b_1	b_2	...
...	...	...	...	...
s^1	g_0			
s^0	h_0			

→ If no sign changes
then system is
stable system

Construction of Routh Array:-

Let the C.E is

$$a_0 s^n + a_1 s^{n-1} + a_2 s^{n-2} + \dots + a_{n-1} s^1 + a_n = 0$$

The coefficients of the polynomial are arranged in two rows as shown below.

$$S^n : a_0 \quad a_2 \quad a_4 \quad \dots$$

$$S^{n-1} : a_1 \quad a_3 \quad a_5 \quad \dots$$

when 'n' is even, the s^n row formed by coefficients of even order terms (i.e. coeff. of even powers of s). and s^{n-1} row formed by coefficients of odd order terms.

when 'n' is odd, s^{th} row

odd " " " "

$s^{\text{th}}-1$ row " " even order terms

The other rows of routh array upto s^0 row can be formed by the following procedure. Each row of routh array is constructed by using the elements of previous two rows.

$$\begin{array}{lcl}
 s^n : & x_0 & x_1 \quad x_2 \quad x_3 \quad x_4 \quad \dots \\
 s^{n-1} : & y_0 & y_1 \quad y_2 \quad y_3 \quad y_4 \quad \dots
 \end{array}$$

Let the next row be,

$$s^{n-2} : z_0 \quad z_1 \quad z_2 \quad z_3 \quad z_4 \quad \dots$$

The elements of s^{n-2} row are given by

$$z_0 = \frac{- \begin{vmatrix} x_0 & x_1 \\ y_0 & y_1 \end{vmatrix}}{y_0} = \frac{y_0 x_1 - x_0 y_1}{y_0}$$

$$z_1 = \frac{- \begin{vmatrix} x_0 & x_2 \\ y_0 & y_2 \end{vmatrix}}{y_0} = \frac{y_0 x_2 - x_0 y_2}{y_0}$$

$$z_2 = \frac{y_0 x_3 - x_0 y_3}{y_0} \quad \text{and so on}$$

$$C.E \Rightarrow s^4 + s^3 + 2s^2 + 3s + 1 = 0$$

$$\begin{array}{r|l}
 s^4 & 1 \quad 2 \quad 1 \\
 s^3 & 1 \quad 3 \\
 s^2 & \frac{(1 \times 2 - 3 \times 1)}{1} \quad \frac{1 - 0}{1} \\
 s^1 & \\
 s^0 &
 \end{array}$$

In the process of constructing row array the missing terms are considered to be zero.

In the construction of Routh array, we are having the following three cases.

- i) Normal Routh Array (Non zero elements in first column)
- ii) A row of All zeros.
- iii) First element of row is zero but some or other elements are not zero.

Case I: Normal Routh Array.

In this case.

- i) If there is no sign change in the first column of Routh array then all the roots are lying on left half of s -plane and system is stable.
- ii) If there is sign change in the first column of Routh array, then system is unstable and the number of roots lying on Right half of s -plane are equal to number of sign changes. The remaining roots are lying on left half of s -plane.

Using the Routh criterion, determine the stability of the system represented by C.E. $s^4 + 8s^3 + 18s^2 + 16s + 5 = 0$.
 Comment on location of roots of C.E.

Sol:

The characteristic equation is

$$s^4 + 8s^3 + 18s^2 + 16s + 5 = 0$$

s^4	1	18	5
s^3	8	16	-
s^2	$\frac{8 \times 18 - 16 \times 1}{8} = 16$	$\frac{8 \times 5 - 0 \times 1}{8} = 5$	
s^1	$\frac{16 \times 16 - 5 \times 8}{16} = 13.5$		
s^0	$\frac{13 \times 5 \times 5 - 16 \times 0}{13.5} = 5$		

s^4	1	18	5
s^3	8	16	
s^2	16	5	
s^1	13.5		
s^0	5		

First column.

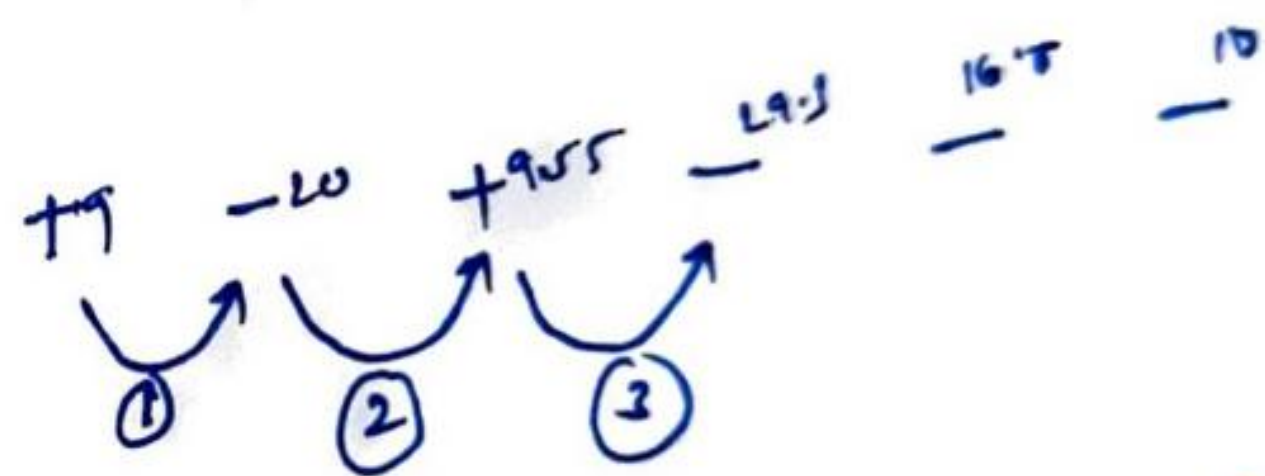
No sign changes in first column

Then system is stable system.

All 4 roots are lying on L-H of s-plane.

$$9s^5 - 20s^4 + 10s^3 - s^2 - 9s - 10 = 0$$

s^5	9	10	-9
s^4	-20	-1	<u>-10</u>
s^3	9.55	-13.5	
s^2	-29.3	-10	
s^1	-16.8		
s^0	-10		



There are 3 - sign changes. system is unstable

3 roots are lying on Right Half s-plane

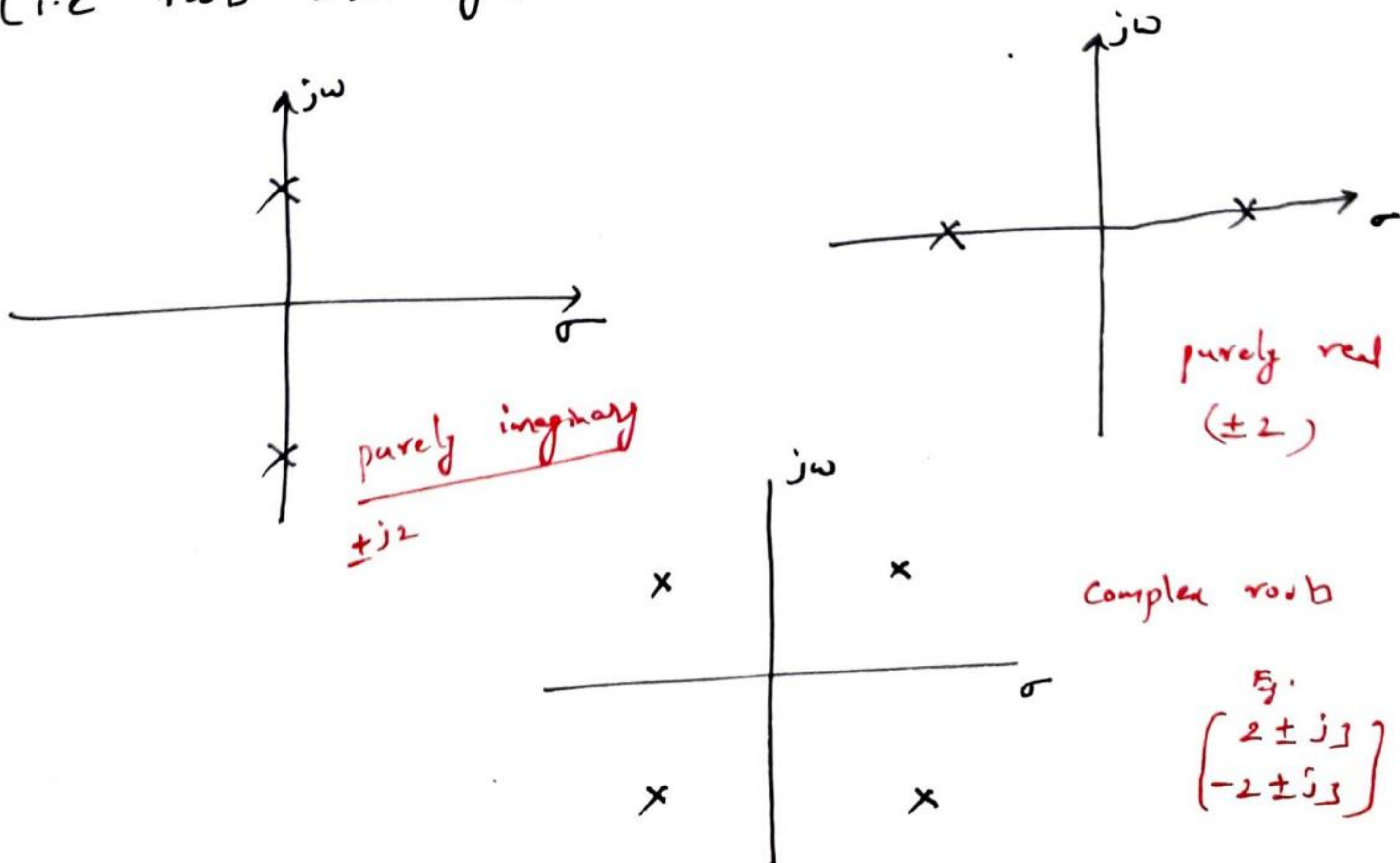
2 roots are " Left " "

Case ii)

Row of all zeros:

An all zero row indicates the existence of even polynomial as a factor of given C.E. This even polynomial factor is called "auxiliary Polynomial". The coefficients of the auxiliary polynomial will always be the elements of the row directly above the row of zeros in Routh array.

The roots of an even polynomial occur in pairs that are equal in ~~size~~ magnitude and opposite in sign. Hence, these can be purely imaginary, purely real (or) complex. purely imaginary and purely real roots occur in pairs. The complex roots occur in group of 4 and these complex roots have quadrantal symmetry. (i.e. roots are symmetrical about real and imag axis)



Method:

step 1) Determine the auxiliary polynomial $A(s)$

step 2) Differentiate the auxiliary polynomial w.r.t 's'

$$\text{i.e. } \frac{dA(s)}{ds}$$

step 3: The row of zero is replaced with coefficients of $\frac{dA(s)}{ds}$.

step 4: Continue the construction of Routh array as usual manner. and the array is interpreted as follows

→ If there are sign changes in the first column of Routh array then the system is unstable. The no. of roots lying on RH of s-plane is equal to the no. of sign changes. The no. of roots on imaginary axis can be estimated from. The roots of auxiliary polynomial. The remaining roots are lying on LH of s-plane.

→ If there are no sign changes in the first column of Routh array then All zero row indicate the existence of purely imaginary roots. and the system is limitedly or marginally stable. The roots auxiliary equation lie on imag. axis. and the remaining roots are lying on left half of s-plane.

$$\rightarrow s^6 + 2s^5 + 8s^4 + 12s^3 + 20s^2 + 16s + 16 = 0$$

Also det. the no. of roots lying in LH, RH and on img. axis plane

s^6	1	8	20	16	
s^5	2	12	16		
s^4	2	12	16		
s^3	0	0	0	0	All zero row.
s^2	6	16			
s^1	2 \cdot 6				
s^0	16				

s^4	$\frac{8 \times 2 - 12 \times 1}{2}$	$\frac{2 \times 20 - 12 \times 16}{2}$	16
s^3	$\frac{2 \times 12 - 2 \times 12}{2}$	$\frac{2 \times 16 - 2 \times 16}{2}$	
s^2	8	24	
s^1	$\frac{8 \times 12 - 24 \times 2}{8}$	16	
s^0	$\frac{24 \times 6 - 16 \times 8}{6}$		

Auxiliary polynomial is $A(s) = 2s^4 + 12s^2 + 16$

$$\frac{dA(s)}{ds} = 8s^3 + 24s$$

It is observed that there is no sign change in the first column of roots array. The row with all zero row indicates that there is a possibility of imaginary roots. Hence the

System is Limitedly (or) Marginally stable.

The Auxiliary polynomial is

$$2s^4 + 12s^2 + 16 = 0$$

$$\text{Let } s^2 = x$$

$$2x^2 + 12x + 16 = 0$$

$$ax^2 + bx + c = 0$$
$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{-12 \pm \sqrt{(12)^2 - 4(16)(2)}}{2 \times 2}$$

$$= \frac{-12 \pm 4}{4}$$

$$= -3 \pm 1 = -2 \text{ (or) } -4$$

$$\begin{array}{l|l} x = -2 & x = -4 \\ s^2 = -2 & s^2 = -4 \\ s = \pm \sqrt{2}j & s = \pm 2j \end{array}$$

Hence Four Roots are lying on imaginary axis.
and Remaining two Roots are lying on Left ^{half} of s-plane.

$$s^7 + 9s^6 + 24s^5 + 24s^4 + 24s^2 + 23s + 15 = 0$$

$$s^7 + 9s^6 + 24s^5 + 24s^4 + 24s^3 + 24s^2 + 23s + 15 = 0$$

s^7	1	24	24	23
s^6	9	24	24	15
s^5	21	21	21	
s^4	5	5	5	
s^3	0	0		
s^2	2.5	5		
s^1	0 -30			
s^0	5			

$$A(s) = 5s^4 + 5s^2 + 5$$

$$\frac{dA(s)}{ds} \Rightarrow 20s^3 + 10s$$

There are two sign changes in first column of the roots array. System is unstable. Two poles are lying on R.H of s-plane.

$$A.E \Rightarrow 5s^4 + 5s^2 + 5 = 0$$

$$\text{Let } s^2 = x$$

$$5x^2 + 5x + 5 = 0$$

$$x^2 + x + 1 = 0$$

$$x = -\frac{1}{2} \pm j\frac{\sqrt{3}}{2}$$

$$\text{But } s^2 = x \Rightarrow s = \sqrt{x} \Rightarrow s = \pm(0.5 + j0.866)$$

Remaining '5' poles are lying on R.H of s-plane.

+	—
+	—
+	
+	
+	
+	
+	

stable system

s^6	+	
s^5	+	
s^4	+	
s^3	+	) ①
s^2	—	) ②
s^1	+	) ③
s^0	—	) ④

unstable system

poly R.H.S $\rightarrow$ 4
L.H.S $\rightarrow$ 2

s^7	+	
s^6	+	
s^5	+	
s^4	+	
s^3	0	0
s^2	+	
s^1	+	
s^0	+	

System is Marginally stable,
4 - poles $\rightarrow$ imag.
3 - $\rightarrow$ L.H.s plane

s^7	+	
s^6	+	
s^5	+	
s^4	+	
s^3	0	0
s^2	+	
s^1	—	) ①
s^0	+	) ②

R.H.s plane - 2
L.H. - - - 7
unstable

s^7	+	) ①
s^6	—	) ②
s^5	+	) ③
s^4	+	
s^3	0	0
s^2	+	
s^1	+	
s^0	+	

unstable

R.H.s plane - 2
L.H.s imag - 5

Case iii) First element of a row is zero.

while constructing routh array, if a zero is occurs as first element of a row then all the elements of next row will infinite. To overcome that problem. let $0 \rightarrow \epsilon$ and complete the routh array in usual way (as that case - I).

finally let $\epsilon \rightarrow 0$ and determine the values of elements of the array which are function of ϵ . The resultant array is interpreted as follows.

- i) If there is no sign change in the first column of the routh array and if there is no row with all zeroes, then system is stable and all the roots are lying on R.H of s-plane.
- ii) similar to case (i) only.
Sign changes are there $\rightarrow$ system unstable
No. of poles lying on R.H of s-plane = No. of sign changes
- iii) If there is row of zero after letting $\epsilon \rightarrow 0$, then there is a possibility of imaginary roots. That can be calculated by case (2) and system may be unstable or marginally stable.

→ Construct Routh Array and determine the stability of the system represented by C.E

$$s^5 + s^4 + 2s^3 + 2s^2 + 3s + 5 = 0$$

Comment on the location of roots of C.E.

s^5	1	2	3
s^4	1	2	5
s^3	$\phi \rightarrow \epsilon$	-2	
s^2	$\frac{2\epsilon+2}{\epsilon}$	5	
s^1	$\frac{-2\left[\frac{2\epsilon+2}{\epsilon}\right] - 5\epsilon}{\frac{2\epsilon+2}{\epsilon}}$		
s^0	5		

$\rightarrow -\frac{(5\epsilon^2 + 4\epsilon + 1)}{2\epsilon + 2}$

on letting $\epsilon \rightarrow 0$ we get

s^5	1	2	3
s^4	1	2	5
s^3	0	-2	
s^2	∞	-2	
s^1	-2		
s^0	5		

Two sign changes

System is unstable

poles lying on R-H of s-plane = 2

" " " " L-H of " = 3

→ Determine the range of 'k' for stability of unity feed back system whose OLTF is

$$G(s) = \frac{k}{s(s+1)(s+2)}$$

Sol:

Routh Array $\Rightarrow$ 

$$\frac{C(s)}{R(s)} = \frac{G(s)}{1 + G(s)H(s)}$$

$\downarrow$ C.E

C.E of CL system $\Rightarrow 1 + G(s) \cdot H(s) = 0$

Given that it is unity f/b
 $H(s) = 1$

$$\Rightarrow 1 + G(s) = 0$$

$$\Rightarrow 1 + \frac{k}{s(s+1)(s+2)} = 0$$

$$\Rightarrow \frac{s(s+1)(s+2) + k}{s(s+1)(s+2)} = 0$$

$$\Rightarrow s(s+1)(s+2) + k = 0$$

$$\Rightarrow s(s^2 + 3s + 2) + k = 0$$

$$\Rightarrow \boxed{s^3 + 3s^2 + 2s + k = 0}$$

$\downarrow$
characteristic equation

$$s^3 + 3s^2 + 2s + k = 0$$

s^3	1	2
s^2	3	k
s^1	$\frac{6-k}{3}$	
s^0	k	

For system to be stable there should be no sign changes in the first column of roth array.

ie $\frac{6-k}{3} > 0 \quad \& \quad k > 0$

$$6-k > 0 \quad \& \quad k > 0$$

$$k < 6 \quad \& \quad k > 0$$

$0 < k < 6$

↙

Range of 'k' value
for system to be stable.

The OLTF of a unity feed back system is given by

$$G(s) = \frac{K}{(s+2)(s+4)(s^2+6s+25)}$$

By applying R-H (Routh-Hurwitz) criterion, discuss stability of CL system as a function of 'K'. Determine the value 'K' which will cause sustained oscillation in the CL system. What are the corresponding oscillating frequencies?

Sol:

The characteristic equation is

$$1 + G(s) \cdot H(s) = 0$$

$$\text{But } H(s) = 1 \quad (\text{given unity feedback})$$

$$1 + G(s) = 0$$

$$1 + \frac{K}{(s+2)(s+4)(s^2+6s+25)} = 0$$

$$(s+2)(s+4)(s^2+6s+25) + K = 0$$

$$\Rightarrow (s^2+6s+8)(s^2+6s+25) + K = 0$$

$$\Rightarrow s^4 + 12s^3 + 69s^2 + 198s + 200 + K = 0$$

s^4	1	69	$200+K$
s^3	12	198	
s^2	52.5	$200+K$	
s^1	$\frac{7995-12K}{52.5}$		
s^0	$200+K$		

For system to be stable

$$\frac{7995-12K}{52.5} > 0 \quad \& \quad 200+K > 0$$

$$12K < 7995 \quad \& \quad K > -200$$

$$K < 666.25 \quad \& \quad K > -200$$

$$\boxed{-200 < K < 666.25}$$

$$0 < K < 666.25 \quad \checkmark$$

Marginally stable system produces sustained oscillations.

in the system. If a row of all zeros occurs in Routh

array we can ~~confirm~~ system may be marginally stable.

To get row of all zeros equate s^1 row to zero.

i.e

$$\frac{7995-12K}{52.5} = 0$$

$$\boxed{K = 666.25}$$

The auxiliary equation is

$$A(s) \Rightarrow 52.5s^2 + (200 + K) = 0$$

But $K = 666.25$

$$\Rightarrow 52.5s^2 + 200 + 666.25 = 0$$

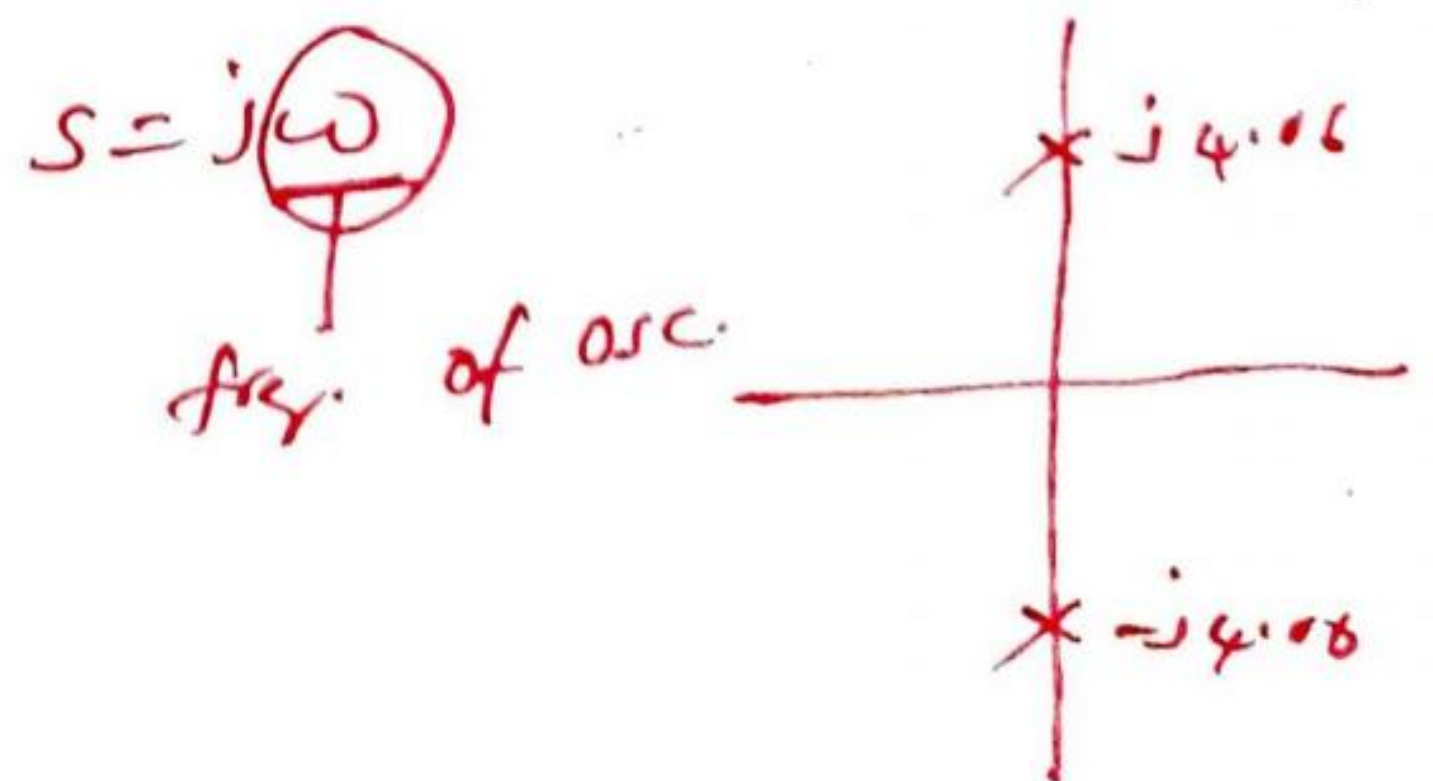
$$\Rightarrow s^2 = -\frac{866.25}{52.5}$$

$$\Rightarrow s^2 = -16.5$$

$$\Rightarrow s = \pm j4.06$$

when $K = 666.25$, the system has roots on imaginary axis and so it oscillates. The frequency of oscillation is given by the value of root on imaginary axis

$\therefore$ The frequency of oscillating, $\boxed{\omega = 4.06 \text{ rad/sec}}$



→ The OLTF of a unity feedback system is given by.

$$G(s) = \frac{k(s+1)}{s^3 + as^2 + 2s + 1}$$

Determine the value of 'k' and 'a' so that the system oscillates at a frequency of 2 rad/sec.

Sol:

$$C.E \Rightarrow 1 + G(s) = 0$$

$$s^3 + as^2 + (2+k)s + 1+k = 0$$

$$\begin{array}{c|cc} s^3 & 1 & 2+k \\ s^2 & a & 1+k \\ s^1 & \frac{a(2+k) - (1+k)}{a} & \\ s^0 & 1+k & \end{array}$$

From s^2 row, the auxiliary polynomial is

$$as^2 + (1+k) = 0 \Rightarrow as^2 = -(1+k)$$

$$s = \pm j \sqrt{\frac{1+k}{a}}$$

But given that, $s = \pm j2$

$$\sqrt{\frac{1+k}{a}} = 2 \Rightarrow \frac{1+k}{a} = 4 \Rightarrow k = 4a - 1 \quad \hookrightarrow \textcircled{1}$$

From 1st row,
$$\frac{a(2+k) - (1+k)}{a} = 0$$

$$\Rightarrow a(2+k) - (1+k) = 0$$

$$\Rightarrow 2a - 1 + k(a - 1) = 0$$

But $k = 4a - 1$ [from ①]

$$\Rightarrow 2a - 1 + (4a - 1)(a - 1) = 0$$

$$\Rightarrow 4a^2 - 3a = 0$$

$$\Rightarrow a(4a - 3) = 0$$

$$a \neq 0, \quad 4a - 3 = 0$$

$$\Rightarrow \boxed{a = 3/4}$$

when $a = 3/4$

$$k = 4a - 1$$

$$\boxed{k = 2}$$

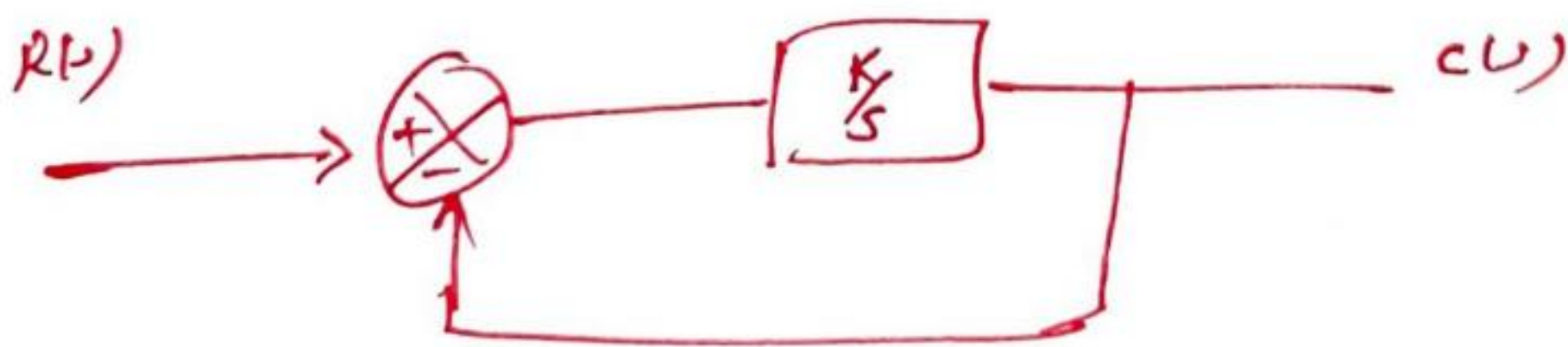
Root Locus:-

- purpose i) To find closed loop system stability.
- ii) To find the range of 'k' value for system stability
- iii) To find K_{mar} , $\omega_{marginal}$
- iv) To find 'k' value for undamped, under damped, critical and over damped systems.
- v) To find Relative stability.

Root $\rightarrow$ roots of $\underbrace{C.E}_{\text{closed loop poly.}}$

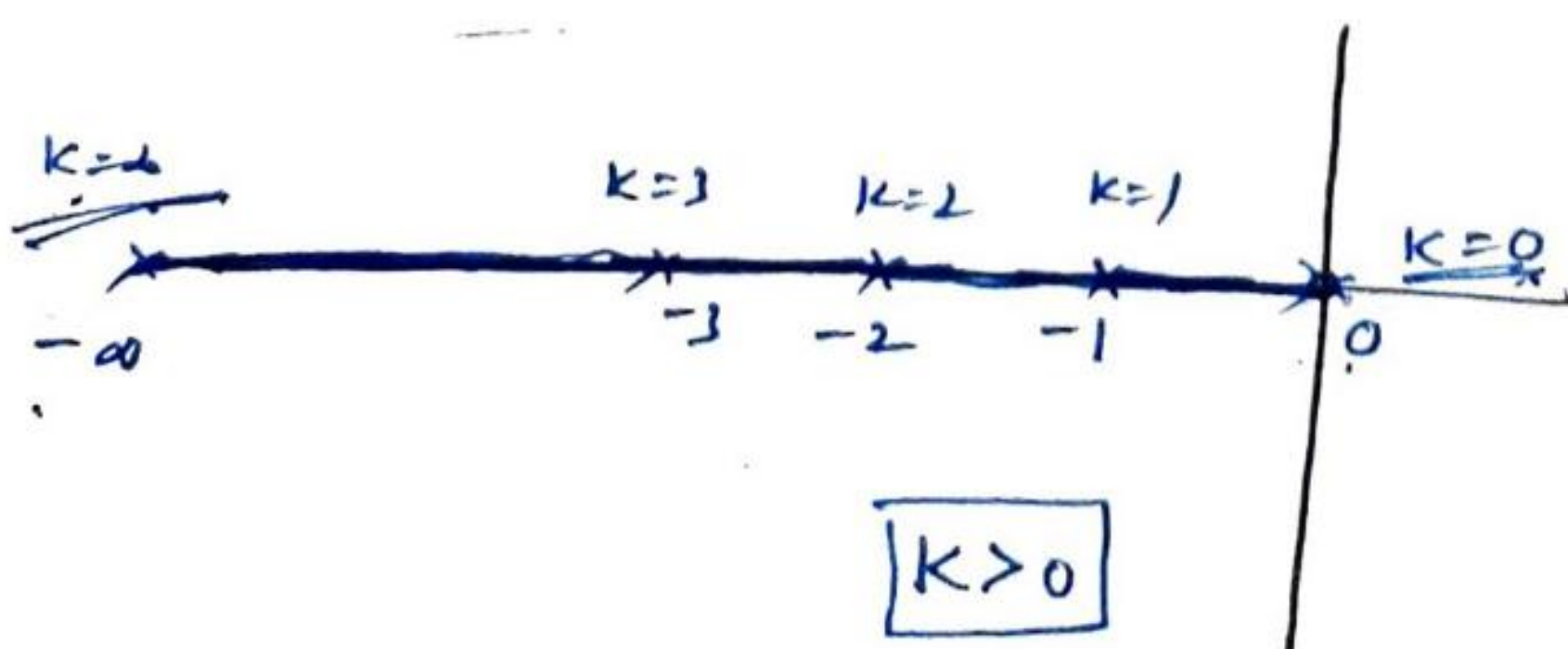
Locus $\rightarrow$ path

Root locus is nothing but closed loop poles path by varying 'k' from 0 to ∞

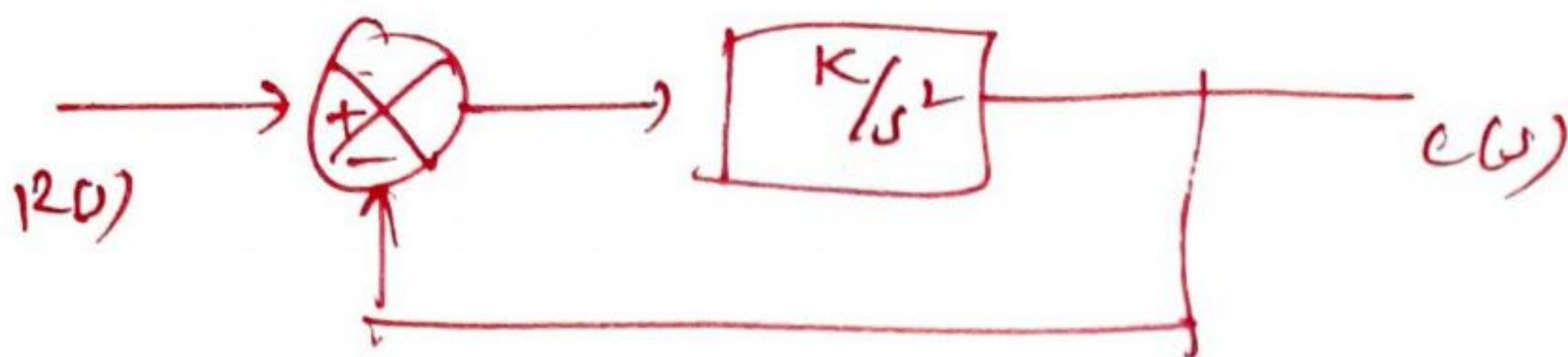


$$\frac{C(s)}{R(s)} = \frac{K/s}{1 + K/s} = \frac{K}{s + K}$$

$$C.E \Rightarrow s + K = 0$$
$$\boxed{s = -K} \rightarrow \text{closed loop poles.}$$



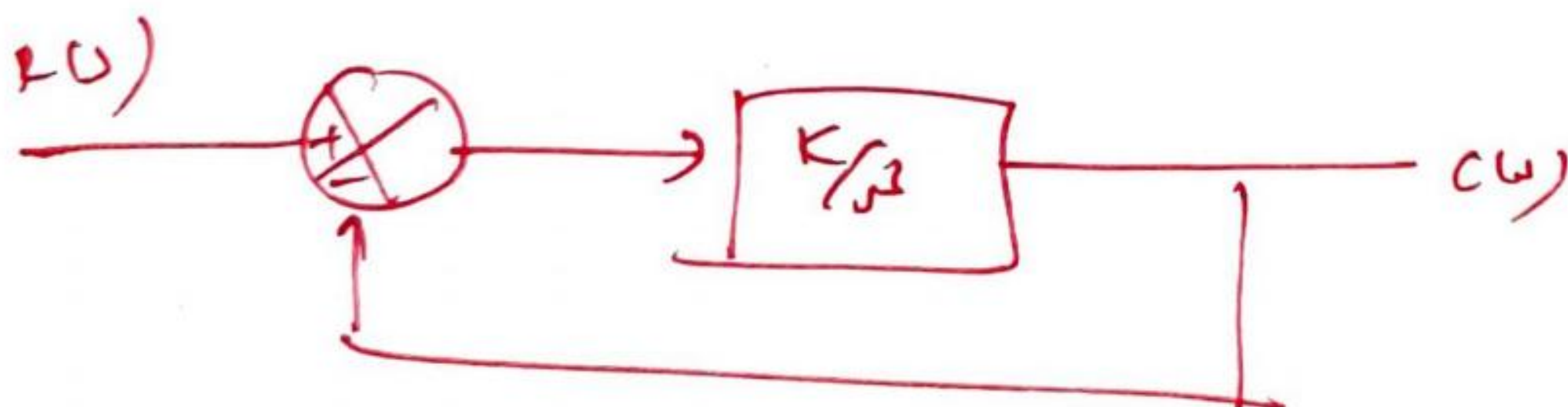
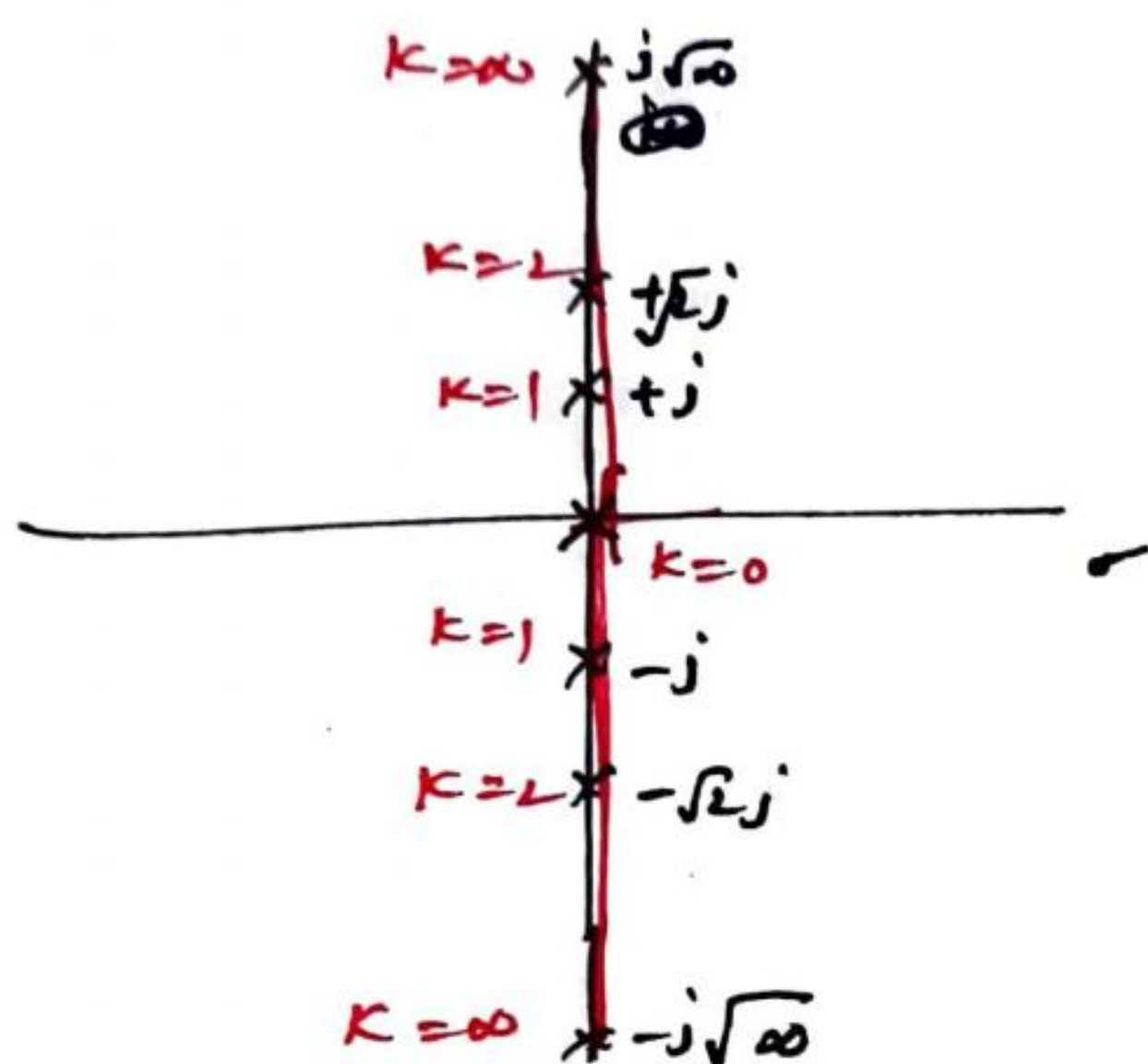
$K = -1$	$\Rightarrow s = 1$
$K = -2$	$\Rightarrow s = 2$



$$\frac{C(s)}{R(s)} = \frac{K/s^2}{1 + K/s^2} = \frac{K}{s^2 + K}$$

$$s^2 + K = 0$$

$$s = \pm j\sqrt{K}$$



$$\frac{C(s)}{R(s)} = \frac{K}{s^3 + K}$$

$$s^3 + K = 0$$

$$s = (-K)^{1/3}$$

→ As order is increased, finding of roots are very difficult.

Hence we cannot draw root locus to the C.F.

→ To draw a root locus diagram, we use open loop T-F

but the analysis is for closed loop systems.

Relationship between CLTF poles and OLTF poles and zeros:

consider an open loop transfer function (OLTF)

$$G(s)H(s) = \frac{K \cdot N(s)}{D(s)}$$

$N(s) \rightarrow$ Numerator polynomial of OLTF
 $D(s) \rightarrow$ Denominator "

$$\text{open loop poles} \Rightarrow D(s) = 0$$

$$\text{open loop zeros} \Rightarrow N(s) = 0$$

Closed loop poles are given by C.E

$$C.E \Rightarrow 1 + G(s) \cdot H(s) = 0$$

$$1 + K \cdot \frac{N(s)}{D(s)} = 0$$

$$\Rightarrow D(s) + KN(s) = 0 \rightarrow \textcircled{1}$$

$\hookrightarrow$ closed loop poles.

Case (i) $K=0$

$$C.E \Rightarrow D(s) = 0 \Rightarrow \text{CL poles}$$

when $K=0$

$$\boxed{\text{CL pole} = \text{OL pole}}$$

Case (ii) $K=\infty$

$$C.E \Rightarrow D(s) + \left(\frac{1}{0}\right)N(s) = 0 \Rightarrow N(s) = 0 \Rightarrow \text{CL poles}$$

when $K=\infty$

$$\boxed{\text{CL pole} = \text{open loop zeros}}$$

$\rightarrow$ The root locus diagram should be started at open loop pole where $K=0$, and it ends at open loop zero where $K=\infty$

Eg:

$$G(s)H(s) = \frac{K(s+1)}{s(s+5)(s+10)}$$

Find where root locus starts and ends.

$$\text{OL poly} \Rightarrow s = 0, -5, -10$$

$$\text{OL zeros} \Rightarrow s = -1$$

$$\text{when } K=0 \Rightarrow \text{RL starts at OL poly} \Rightarrow 0, -5, -10$$

$$K=\infty \Rightarrow \text{" ends " OL zeros} \Rightarrow -1, \infty, \infty$$

→ In root locus diagram, the no. of poles must be equal to the number of zeros. If zeros are less, then we assume the zeros at infinity.

Angle and Magnitude Conditions:

Angle conditions

Consider C.E

$$\text{i.e. } 1 + G(s)H(s) = 0$$

$$G(s)H(s) = -1 + j0$$

$\underline{a+jb}$	angle = $+\tan^{-1}(b/a)$ magnitude = $\sqrt{a^2+b^2}$
$\frac{1}{a+jb}$	angle = $-\tan^{-1}(b/a)$ magnitude = $\frac{1}{\sqrt{a^2+b^2}}$

$$\angle G(s)H(s) = \angle -1 + j0$$

$$= \pm 180^\circ$$

$$= \pm 1(180^\circ)$$

Purpose of angle condition:

To check the any point lies on the root locus or not. that means all points on Root locus satisfies the angle condition.

Eg:

$$G(s)H(s) = \frac{K}{s(s+2)(s+5)}$$

check whether the following points on RL or not

i) $s = -1$

ii) $s = -4$

$$\begin{aligned} \angle G(s)H(s) \Big|_{s=-1} &= \frac{\angle K}{\angle -1 \angle -2 \angle -5} \\ &= \frac{0^\circ}{\pm 180^\circ \ 0^\circ \ 0^\circ} \end{aligned}$$

$$\begin{aligned} \underline{K} \quad a+jb \\ a=K, \ b=0 \\ \theta = \tan^{-1}(b/a) = 0^\circ \end{aligned}$$

$$= \mp 180^\circ \rightarrow \text{odd multiple of } 180^\circ$$

$s = -1$ point lies on Root locus.

$$\left| \frac{G(s)H(s)}{s+4} \right| = \frac{LK}{L-4 \quad L-2 \quad L}$$

$$= \frac{0^\circ}{\pm 180^\circ \pm 180^\circ \quad 0^\circ}$$

$$= \mp 360^\circ \rightarrow \text{even multiple of } 180^\circ$$

It does not satisfy the angle condition. Given point is not lying on Root locus.

Magnitude condition:

$$1 + G(s)H(s) = 0$$

$$G(s)H(s) = -1 + j0$$

$$M = |G(s)H(s)| = 1$$

at any point
which is on RL

purpose: To find system gain at point which is on Root locus

Eg: Find the system gain 'K' at a point $s = -2 + j2$. to the given system $G(s)H(s) = \frac{K}{s(s+4)}$

Sol:

$$\left| \frac{G(s)H(s)}{s+4} \right| = \frac{|K|}{|(-2+j2)| |2+j2|}$$

$s = -2 + j2$

$$1 = \frac{K}{\sqrt{4+4} \sqrt{4+4}}$$

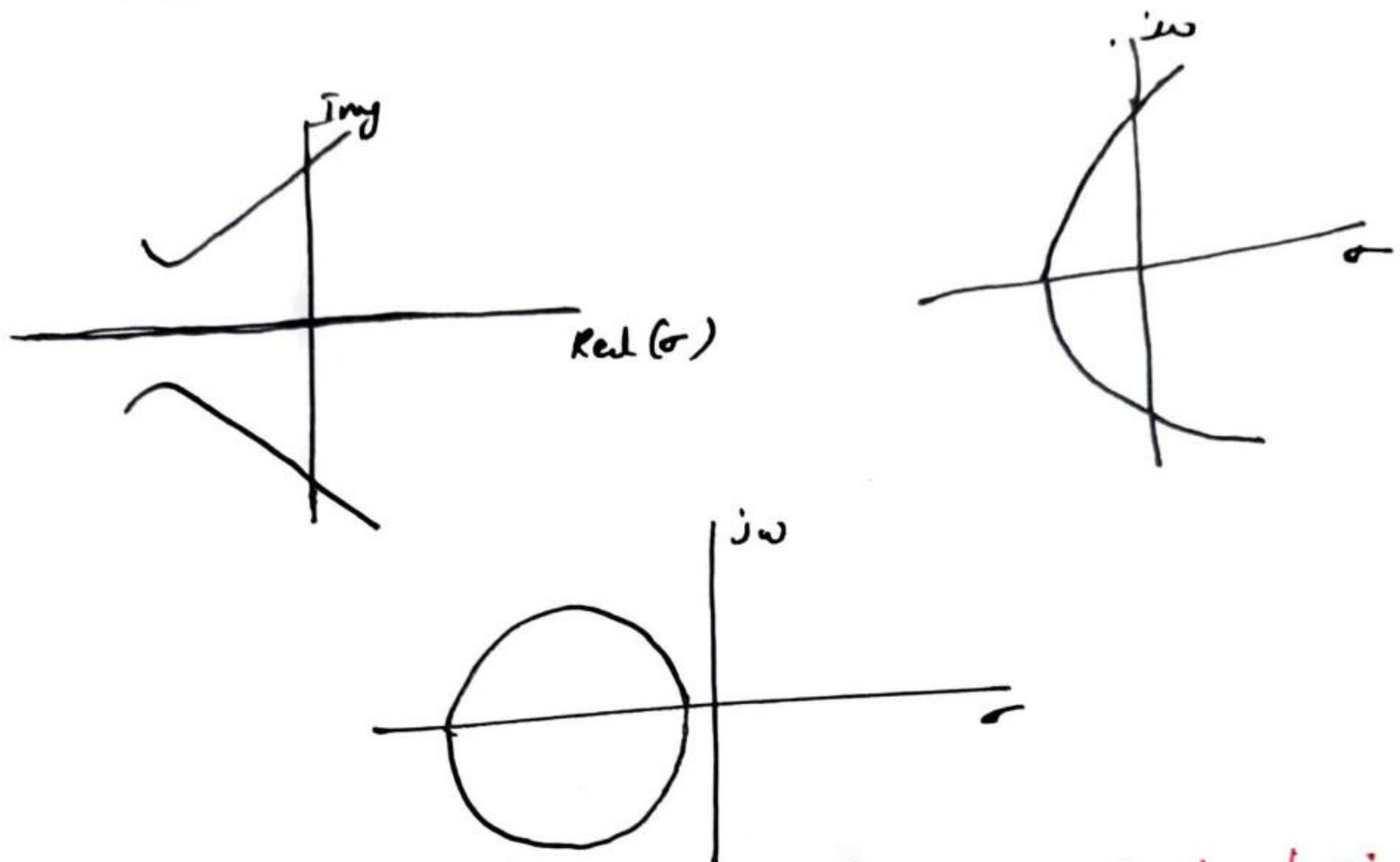
$$\frac{K}{\sqrt{8} \sqrt{8}} = 1$$

$$\boxed{K = 8}$$

Construction Rules for Root Locus:-

Rule 1: The Root locus diagrams are symmetric about the real axis.

Eg:



Rule 2: No. of Root Locus branches / No. of Root loci

Each branch of Root Locus originates from open loop pole where $K=0$ and terminates at either on a finite open loop zero (or open loop zero at infinity) where $K=\infty$.

The no. of Root Locus branches completely depends on no. of poles and zeros.

i) If $P > Z$, then no. of root loci = No. of poles

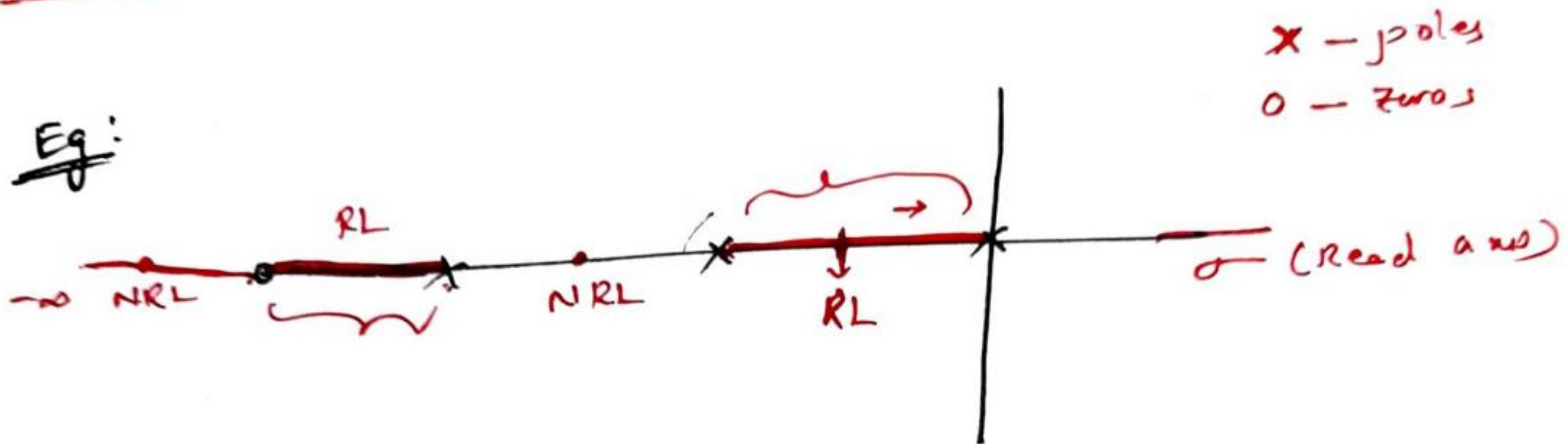
ii) If $P < Z$, " " " = No. of Zeros

where P = No. of poles
 Z = No. of zeros.

Rules: Read axis Root locy branches

A point to exist on real axis root locy branch then the sum of the poles and zeros to the right side of that point should be odd number.

Eg:

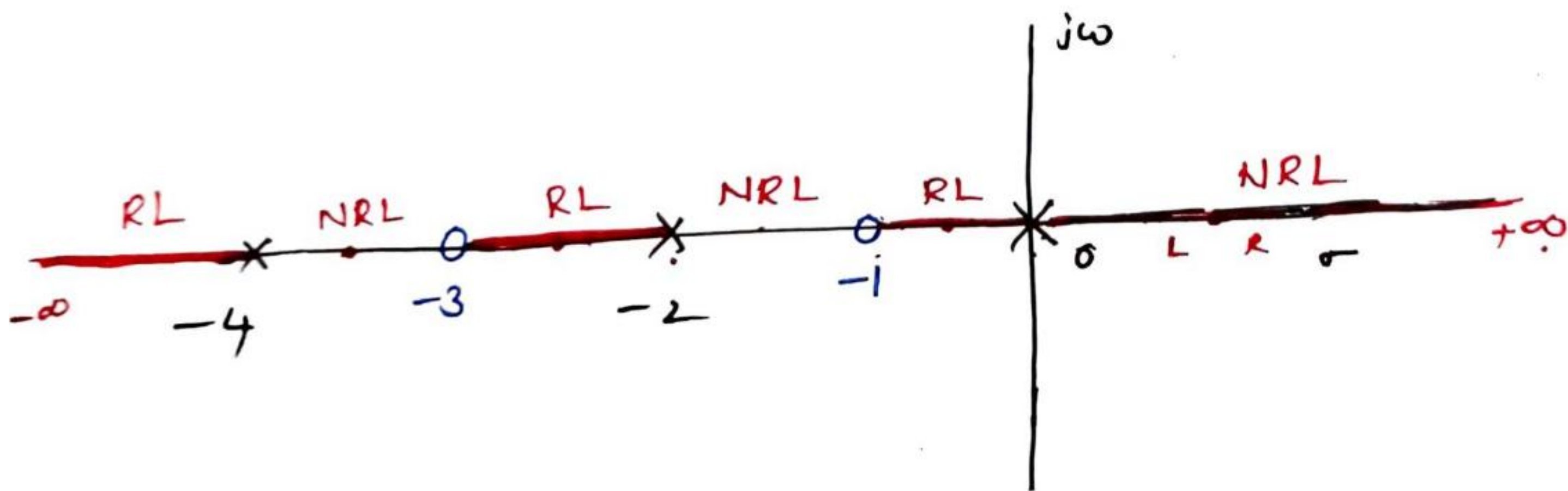


Eg:

$$G(s)H(s) = \frac{K(s+1)(s+3)}{s(s+2)(s+4)}$$

poles $\rightarrow s = 0, -2, -4$

zeros $\rightarrow s = -1, s = -3$



Rule 4: Angle of Asymptotes:

The root branches that tend to infinity, do so along straight line asymptotes making angle with the real axis is given by.

$$\text{Angle of Asymptotes } \theta = \frac{(2q+1)180^\circ}{P-Z}$$

where $q = 0, 1, 2, \dots, \frac{P-Z-1}{2}$ upper limit

$$\text{The No. of asymptotes} = P-Z$$

where P is No. of poles of OLTF
 Z " " " " " " " " " " " "

Rule 5: Centroid:

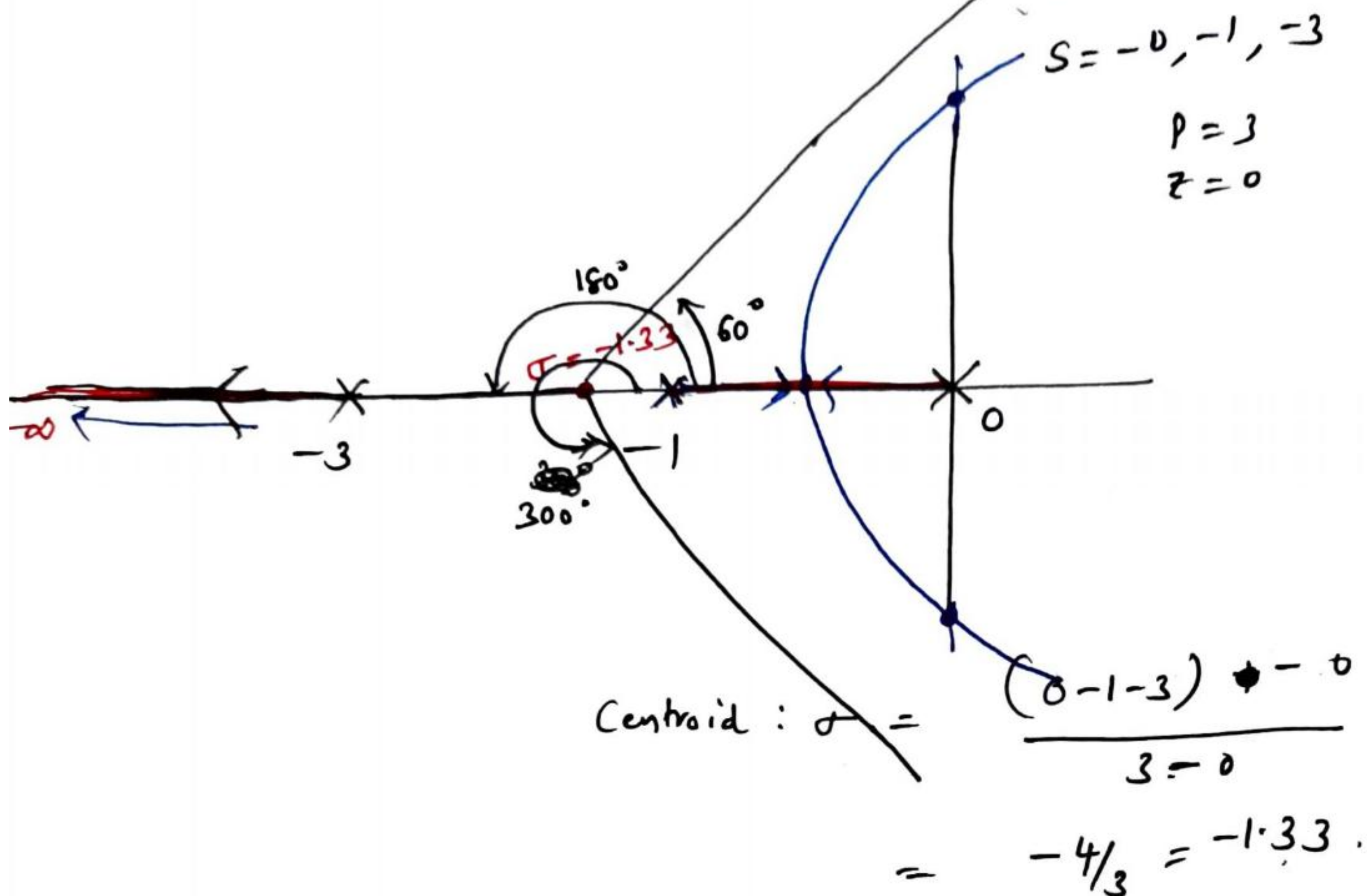
The point of intersection of the asymptotes with the real axis at $s = -\sigma_A$ where.

$$\text{Centroid } (\sigma) = \frac{\sum \text{Real part of poles} - \sum \text{Real part of zeros}}{P-Z}$$
$$= \frac{\sum \text{sum of poles} - \sum \text{sum of zeros}}{P-Z}$$

The centroid may be located anywhere on the real axis. It may or may not be on root locus branch.

Eg:

$$G(s)H(s) = \frac{K}{s(s+1)(s+3)}$$



$$\text{angle of asymptotes } \theta = \frac{(2q+1)180^\circ}{p-z}$$

$$\text{when } q = 0, 1, \dots, p-z-1$$

$$= 0, 1, \dots, \frac{3-0-1}{2}$$

$$= 0, 1, 2$$

$$\begin{aligned} \text{At } q=0 & \Rightarrow \theta_1 = \frac{180^\circ}{3} = 60^\circ \\ q=1 & \Rightarrow \theta_2 = \frac{3 \times 180^\circ}{3} = 180^\circ \\ q=2 & \Rightarrow \theta_3 = \frac{5 \times 180^\circ}{3} = 300^\circ \end{aligned}$$

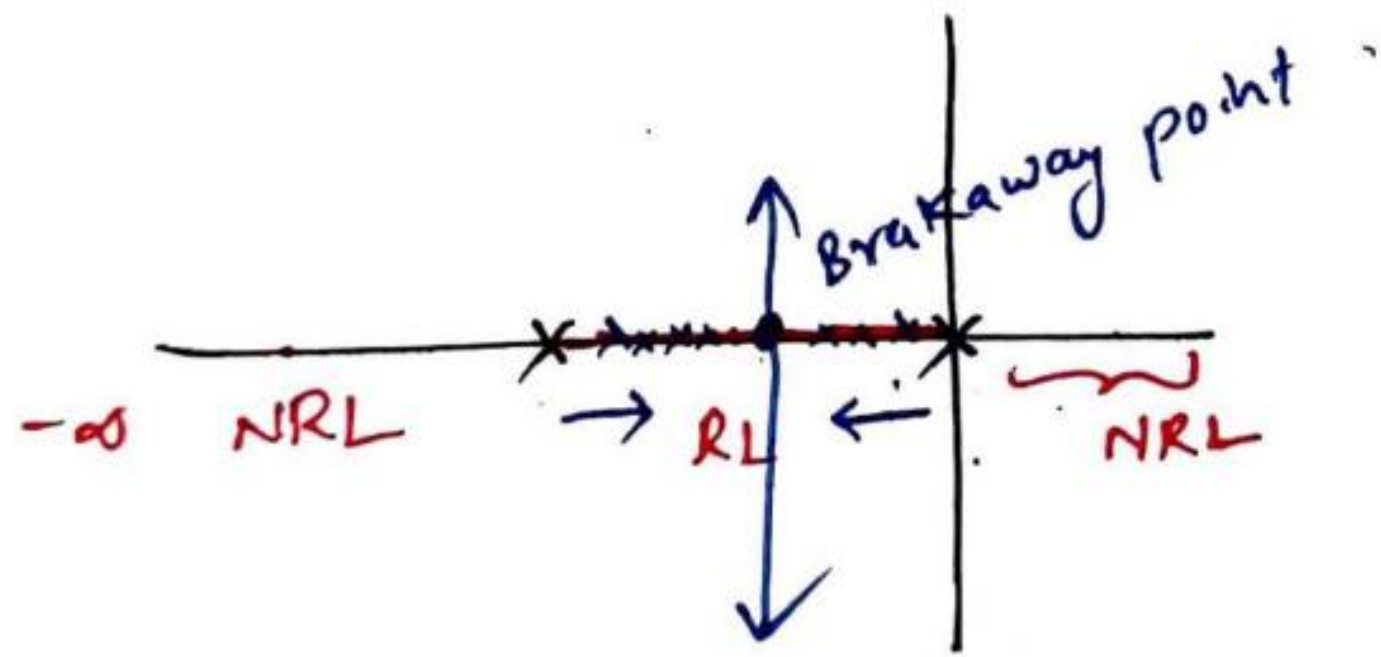
Rule 6 Break points

It is nothing but the meeting point of two poles or zeros. Break points are two types

- i) Break away point
- ii) Break in point

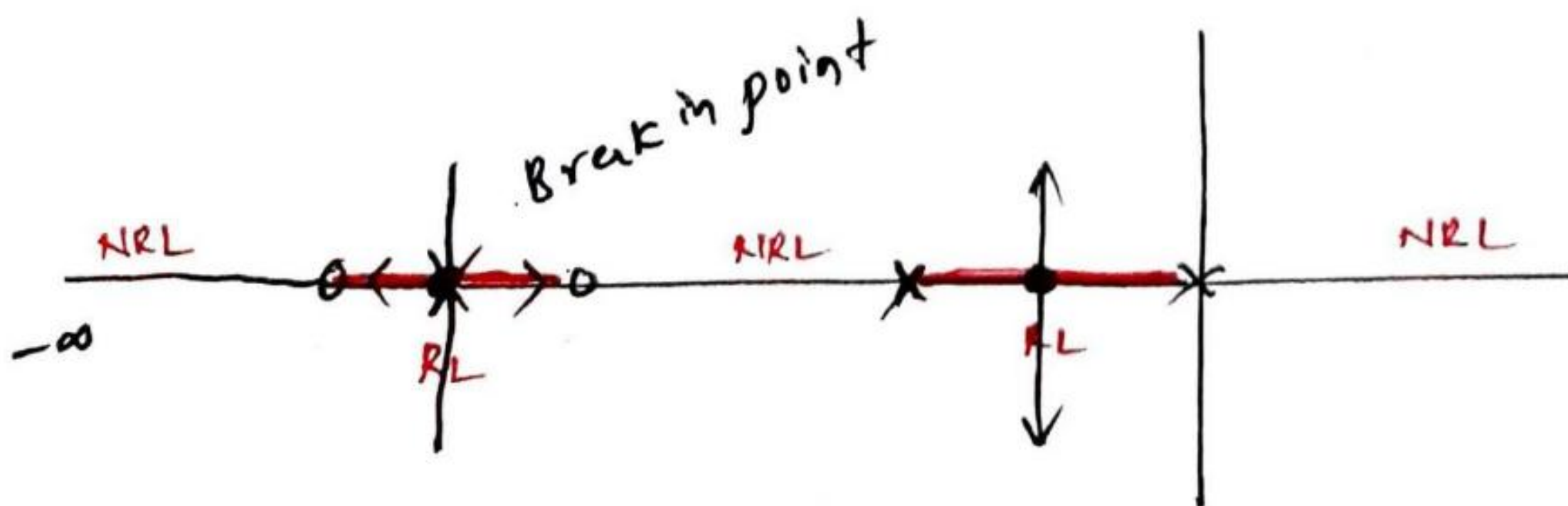
Break away point:

It is the point at which the root locus branches leave the real axis. Then it is called break away points. Mostly it will exist in b/w two poles.



Break in point: The point at which the root locus branches enter into the real axis. Then it is called break in point.

The root locus branches leaving (or) entering the real axis with an angle of $\pm \frac{180^\circ}{n}$, where 'n' is no. of poles meet at the break point.



Finding of location of break points:-

step 1: Replace $G(s)H(s)$ by -1

$$\begin{cases} 1 + G(s)H(s) = 0 \\ G(s)H(s) = -1 \end{cases}$$

- 2) Rearrange above equation in the form of $K = f(s)$
- 3) Differentiate $-K$ w.r.t s and make it equal to ∞ or 0

$$\frac{dK}{ds} = 0$$

- 4) The roots of $\frac{dK}{ds} = 0$ gives valid and invalid breakpoints.
- 5) The valid break point is the one which must lie on root locus. for which the K value must be positive

Rule 7: Intersection point with an Imaginary axis

It can be find by using R-H criterion.

procedure:

- i) write the C.E
- ii) Construct Routh Array
- iii) Find $K_{marginal}$.
- iv) From A.E, find intersection points.

Sketch the root locus of the system where

$$\text{OLTF is, } G(s) = \frac{K}{s(s+2)(s+4)}$$

Sol:

Step 1: To locate poles and zeros.

The poles of OLTF are the roots of the equation

$$s(s+2)(s+4) = 0$$

The poles are lying at $s = 0, -2, -4$

Hence $P_1 = 0, P_2 = -2, P_3 = -4$

The poles are marked by X (cross) in s-plane.

Step 2: To find the root locus on real axis.

choose a test point on real axis b/w $s=0$ and $s=-2$

To the right side of this point, the total number of real poles and zeros is one, which is odd number. Hence the real axis

b/w $s=0$ and $s=-2$ be a part of root locus.

choose " " b/w $s=-2$ & $s=-4$ " "

" " are two which is even. Hence " " $s=-2$ & $s=-4$ will not be a part of root locus.

choose test point " , $s=-4$ & $s=-\infty$ " ,

" " three which is odd number. Hence " ,

$s=-4$ & $s=-\infty$ will be on root locus.

Step 1: To find asymptotes and Centroid:

Since there are three poles and there is no finite zero. Hence all these poles root locus branches end at zeros at infinity.

$$\text{Angles of Asymptotes } \theta = \frac{180^\circ(2q+1)}{p-z}$$

$$\text{where } q = 0, 1, \dots, p-z-1$$

$$\text{But } \left. \begin{array}{l} p=3 \\ z=0 \end{array} \right\} \quad \begin{array}{l} q = 0, 1, \dots, \underline{3-0-1} \\ \quad = 0, 1, 2 \end{array}$$

$$\text{when } q=0, \theta_1 = \frac{180^\circ(1)}{3} = 60^\circ$$

$$q=1, \theta_2 = \frac{180^\circ(3)}{3} = 180^\circ$$

$$q=2, \theta_3 = \frac{180^\circ \times 5}{3} = 300^\circ$$

$$\text{Centroid } (\sigma) = \frac{\text{Sum of poles} - \text{Sum of zeros}}{p-z}$$

$$= \frac{(0-2-4) - 0}{3-0}$$

$$= -\frac{6}{3} = -2$$

4) To find the break away point.

$$\text{Consider C.E} \Rightarrow 1 + G(s)H(s) = 0$$

$$\Rightarrow 1 + \frac{K}{s(s+2)(s+4)} = 0$$

$$\Rightarrow s(s+2)(s+4) + K = 0$$

$$\Rightarrow s(s^2 + 6s + 8) + K = 0$$

$$\Rightarrow K = -(s^3 + 6s^2 + 8s)$$

$$\boxed{K = f(s)}$$

Diff. above eqn w.r.t 's' & make it equal to 0

$$\frac{dK}{ds} = 0$$

$$-(3s^2 + 12s + 8) = 0$$

$$3s^2 + 12s + 8 = 0$$

$$s = \frac{-12 \pm \sqrt{12^2 - 4 \times 3 \times 8}}{2 \times 3} = \frac{-12 \pm \sqrt{144 - 96}}{6} = \frac{-12 \pm \sqrt{48}}{6}$$

(d)
-0.845 ✓
-3.154 ✗

Check for k:

$$K = -(s^3 + 6s^2 + 8s)$$

when $s = -0.845 \rightarrow \text{valid}$

$$K = -[(0.845)^3 + 6(0.845)^2 + 8(0.845)]$$
$$= 3.08 \cdot (\text{positive})$$

when $s = -3.154, \rightarrow \text{Invalid}$

$$K = -3.08 \cdot (\text{negative})$$

Step 5: To find angle of departure/arrival.

Since there are no complex pole or zero, we need not to find angle of departure or arrival.

Step 6: Intersection point with imaginary axis.

consider C.E $\Rightarrow 1 + G(s)H(s) = 0$

$$s^3 + 6s^2 + 8s + K = 0$$

$$\begin{array}{c|cc} s^3 & 1 & 8 \\ s^2 & 6 & K \\ s^1 & \frac{48-K}{6} & \\ s^0 & K & \end{array}$$

equating s^1 row to 0

$$\frac{48-K}{6} = 0$$

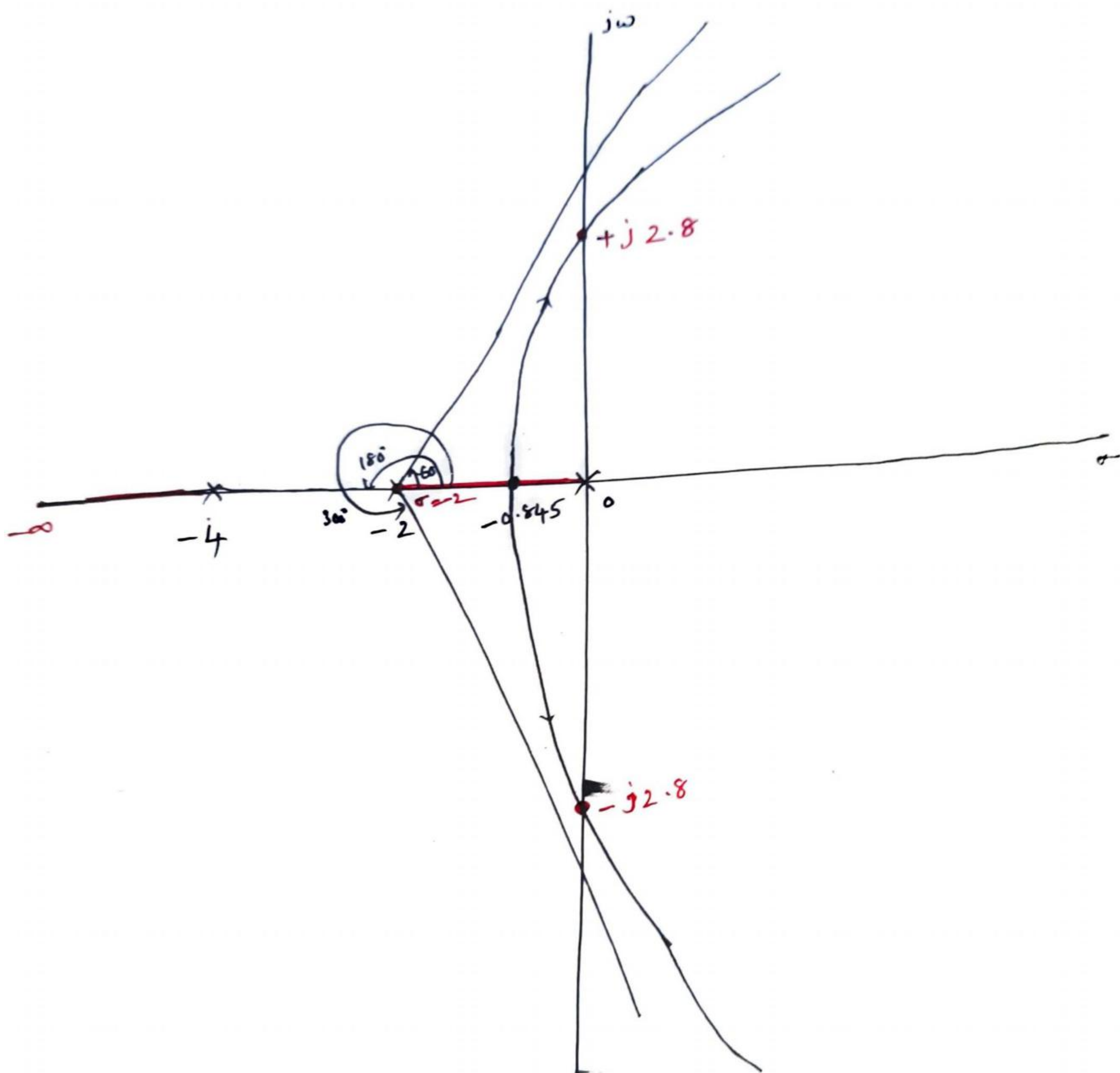
$$\boxed{K = 48}$$

$$\text{A.E} \Rightarrow 6s^2 + K = 0$$

$$6s^2 = -48$$

$$s^2 = -8$$

$$s = \sqrt{-8} = \pm j 2.8$$



The complete root locus ~~branch~~ has 3 branches. one branch starts at the pole $s = -4$ and travel through negative real axis to meet the zero at infinity. The other two root locus branches start at $s = 0$ and $s = -2$ and travel through negative real axis, break away from real axis at $s = -0.845$, then crosses $j\omega$ axis at $s = \pm j2.8$ travel parallel to asymptotes and meet the zeros at infinity.

A unity feedback system has OLTF $G(s) = \frac{K}{s(s^2+4s+13)}$
Sketch the Root Locus.

Sol:

step 1: To locate poles and zeros

The poles of OLTF are roots of $s(s^2+4s+13) = 0$

The roots of $s^2+4s+13 = 0$ are

$$s = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$ax^2+bx+c=0$

$$s = \frac{-4 \pm \sqrt{(4)^2 - 4(1)(13)}}{2(1)}$$
$$= -2 \pm j3$$

The poles are lying at $s=0, -2+j3, -2-j3$

$$P_1 = 0, P_2 = -2+j3, P_3 = -2-j3$$

step 2: To find the root locus on Real axis.

step 3: To find angle of asymptotes and centroid.

$$\text{centroid } \sigma = \frac{\sum \text{poles} - \sum \text{zeros}}{P-Z}$$
$$= \frac{0 - 2+j3 - 2-j3 - 0}{3-0}$$
$$= -4/3$$
$$= -1.33$$

$$\text{angle of asymptotes } \theta = \frac{(2q+1)180^\circ}{p-z}$$

$$\text{where } q = 0, 1, \dots, p-z-1$$

$$= 0, 1, \dots, \underbrace{3-0-1}_2$$

$$= 0, 1, 2$$

$$\text{when } q=0$$

$$\theta_1 = \frac{180^\circ}{3} = 60^\circ$$

$$\text{when } q=1$$

$$\theta_2 = \frac{3 \times 180^\circ}{3} = 180^\circ$$

$$\text{when } q=2$$

$$\theta_3 = \frac{5 \times 180^\circ}{3} = 300^\circ$$

step 4:

To find break points.

$$\text{consider C.E. } 1 + G(s) \cdot H(s) = 0$$

$$1 + \frac{K}{s(s^2 + 4s + 13)} = 0$$

$$\Rightarrow s^3 + 4s^2 + 13s + K = 0$$

$$K = -s^3 - 4s^2 - 13s \longrightarrow \textcircled{1}$$

$$\frac{dK}{ds} = -3s^2 - 8s - 13 = 0$$

$$s = \frac{-8 \pm \sqrt{8^2 - 4(3)(13)}}{2(3)}$$

$$= -1.33 + j1.6$$

check for K:

$$\text{when } s = -1.33 + j1.6$$

$$K = -s^3 - 4s^2 - 13s \neq \text{real and positive}$$

The root locus has neither breakaway nor break in point

step 5- To find the angle of departure:

In given OLT.F, we have complex poles. Hence we have to find angle of departure.

$$\phi_d = 180^\circ - (\sum \phi_p - \sum \phi_z)$$

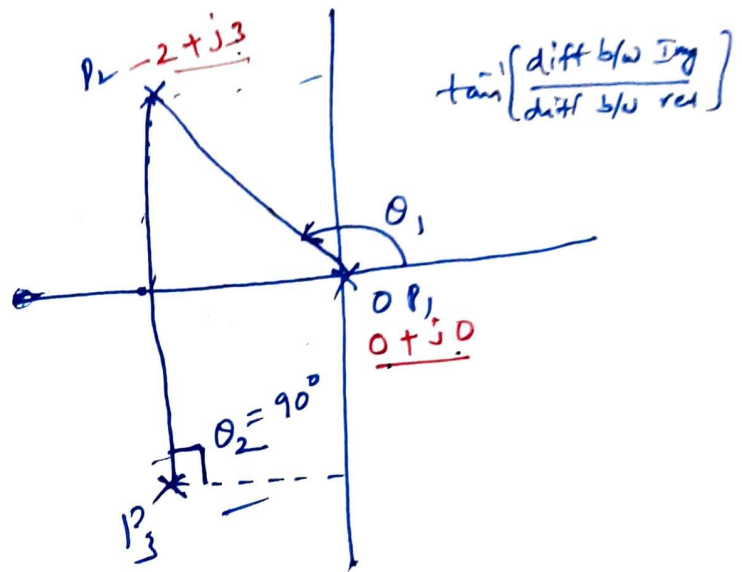
$$\theta_1 = \tan^{-1} \left(\frac{0-3}{0+2} \right)$$

$$= -\tan^{-1} \left[\frac{3}{2} \right]$$

$$= 180^\circ - \tan^{-1} \left[\frac{3}{2} \right]$$

$$= 123.7^\circ$$

$$\theta_2 = 90^\circ$$



$$\phi_d = 180^\circ - [\theta_1 + \theta_2]$$

$$= 180^\circ - [123.7^\circ + 90^\circ]$$

$$= -33.7^\circ$$

The angle of departure at complex pole P_3 is negative of angle of departure at complex pole P_2 .

$$\text{Angle of departure at } P_3 = +33.7^\circ$$

steps:

To find intersecting point with imag axis

Consider C.E

$$1 + G(s)H(s) = 0$$

$$1 + \frac{k}{s(s^2 + 4s + 13)} = 0$$

$$s^3 + 4s^2 + 13s + k = 0$$

$$\begin{array}{c|cc} s^3 & 1 & 13 \\ s^2 & 4 & k \\ s^1 & \frac{52-k}{4} & \\ s^0 & k & \end{array}$$

s^1 row to zero

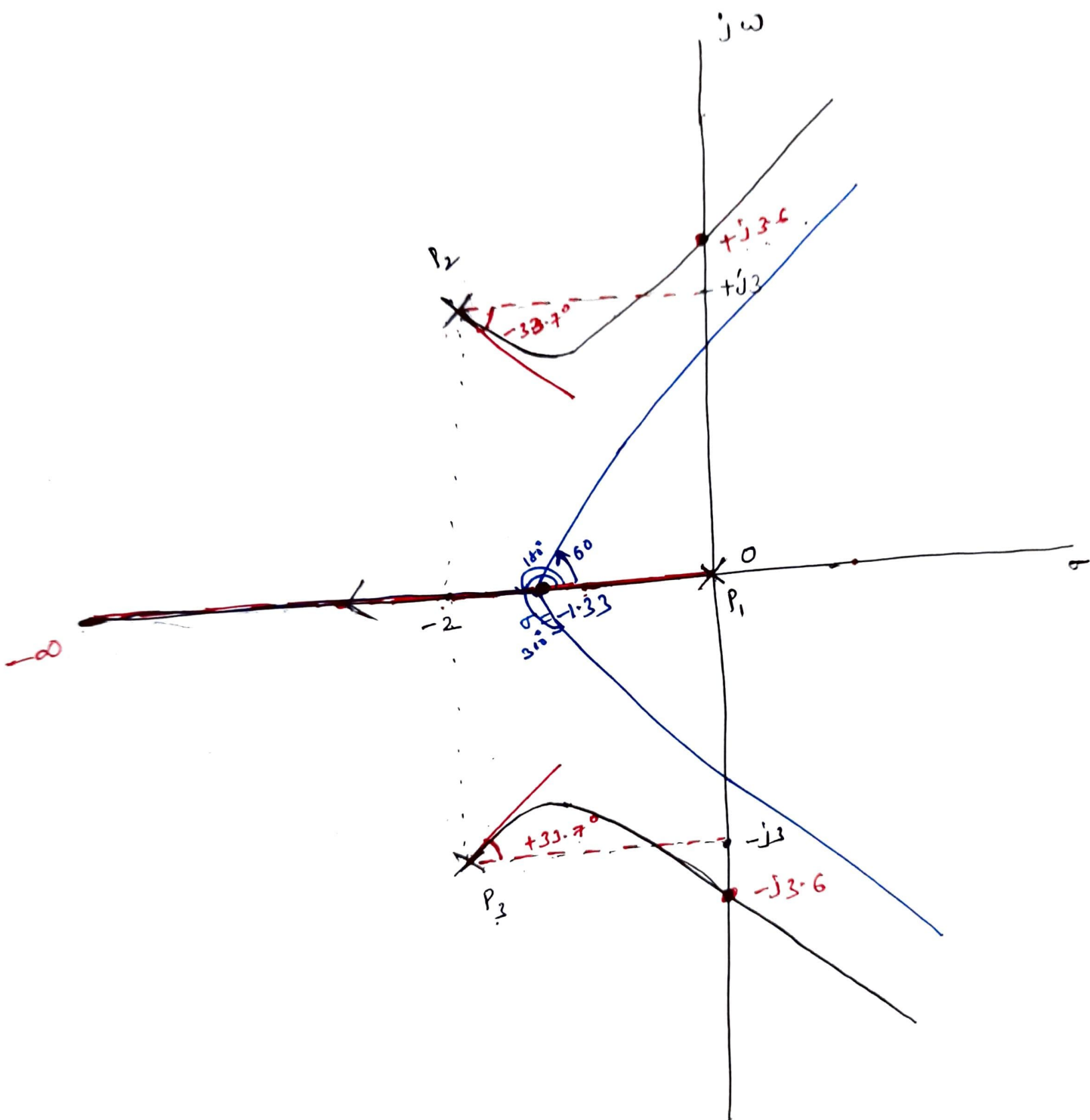
$$\frac{52-k}{4} = 0$$
$$\boxed{k = 52}$$

$$A.E \Rightarrow 4s^2 + k = 0$$

$$4s^2 + 52 = 0$$

$$s^2 = -13$$

$$\boxed{s = \pm j\sqrt{13} = \pm j3.6}$$



→ OLTF $G(s)H(s) = \frac{K(s+9)}{s(s^2+4s+11)}$

sol:
step 1:

zeros $\Rightarrow s+9=0 \Rightarrow s=-9$

poles $\Rightarrow s(s^2+4s+11)=0$

$s=0, s=-2 \pm j2.64$

step 2:

Real axis root locus branches.

step 3:

Centroid $\sigma =$

$$\frac{\sum \text{poles} - \sum \text{zeros}}{p-z}$$

$$= \frac{0 - 2 \pm j2.64 - 2 - j2.64 + 9}{3-1}$$

$$= \frac{+0.5}{2} = 2.5$$

angle of asymptote $\theta = \frac{(2q+1)180^\circ}{p-z}$

$$q = 0, 1, 2 \dots \frac{p-z-1}{3-1-1} = 1$$

$$= 0, 1$$

$q=0$ $\theta_1 = \frac{180^\circ}{3-1} = 90^\circ$

$q=1$ $\theta_2 = \frac{3 \times 180^\circ}{3-1} = 270^\circ$

Step 4:

To find break away and break in points

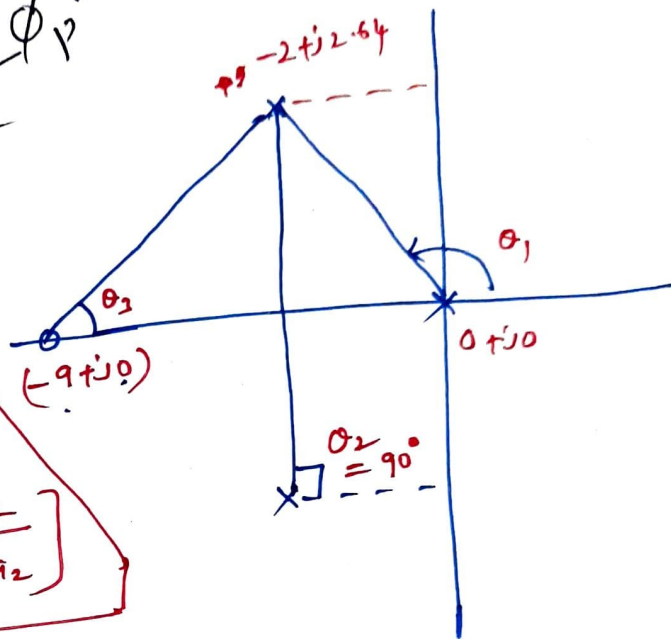
There is no break points.

Step 5:

Angle of departure:

$$\phi_d = 180^\circ - [\sum \phi_p - \sum \phi_z]$$

$$\left[\begin{array}{l} a_2 + jb_2 \\ a_1 + jb_1 \\ \tan^{-1} \left[\frac{b_1 - b_2}{a_1 - a_2} \right] \end{array} \right]$$



$$\begin{aligned} \theta_1 &= \tan^{-1} \left[\frac{0 - 2.64}{0 + 2} \right] \\ &= -\tan^{-1} [1.32] = 180^\circ - \tan^{-1} (1.32) \\ &= 127^\circ \end{aligned}$$

$$\theta_2 = 90^\circ$$

$$\begin{aligned} \theta_3 &= \tan^{-1} \left[\frac{0 - 2.64}{-9 + 2} \right] = \tan^{-1} \left[\frac{2.64}{7} \right] \\ &= 20.7^\circ \end{aligned}$$

$$\phi_d = 180^\circ - [\sum \phi_p - \sum \phi_z]$$

$$= 180^\circ - [\theta_1 + \theta_2 - \theta_3]$$

$$= -16.4^\circ \quad [\text{for } P_2]$$

For pole P_1

$$\boxed{\phi_d = +16.4^\circ}$$

$$\begin{aligned} &-16.4^\circ + 360^\circ \\ &= \underline{343.6} \end{aligned}$$



Intersection point with imaginary axis:

Consider C.E

$$1 + G(s)H(s) = 0$$

$$1 + \frac{K(s+9)}{s(s^2+4s+11)} = 0$$

$$s^3 + 4s^2 + 11s + Ks + 9K = 0$$

$$s^3 + 4s^2 + (11+K)s + 9K = 0$$

$$\begin{array}{c|cc} s^3 & 1 & 11+K \\ s^2 & 4 & 9K \\ s^1 & \frac{44-5K}{4} & \\ s^0 & 9K & \end{array}$$

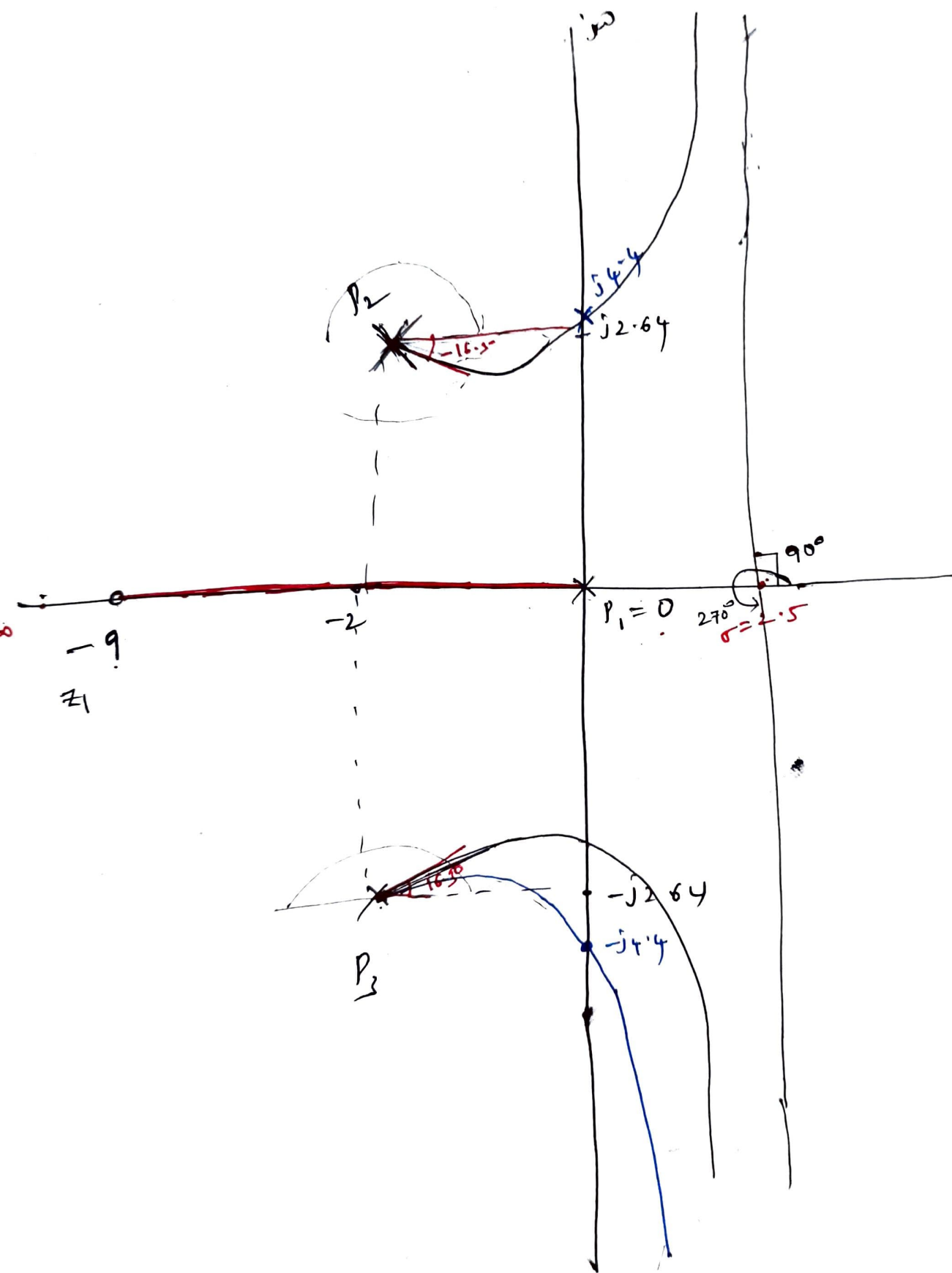
$$\begin{aligned} \frac{44-5K}{4} &= 0 \\ K &= 44/5 \end{aligned}$$

$$A.E \Rightarrow 4s^2 + 9K = 0$$

$$4s^2 + 9\left[\frac{44}{5}\right] = 0$$

$$s = \pm j\sqrt{19.8} = \pm j4.4$$

$$G(s) \cdot H(s) = \frac{K}{s(s+9)(s^2+4s+20)}$$



$$\rightarrow G(s) = \frac{k}{s(s+4)(s^2+4s+20)}$$

Sol: step 1: poles are $s = 0, -4, -2 \pm j4$

No. of poles^(P) = 4.

Centroid: $\sigma = \frac{\sum \text{sum of poles} - \sum \text{sum of zero}}{p - z}$

$$= \frac{0 - 4 - 2 + j4 - 2 - j4 - 0}{4 - 0}$$

$$= \frac{-8}{4} = -2$$

angle of asymptotes:

$$\theta = \frac{(2q+1)180^\circ}{p-z}, \quad q = 0, 1, 2, \dots, p-z-1$$

$$= 0, 1, 2, 3$$

when $\underline{q=0}$ $\theta_1 = \frac{180^\circ}{4} = 45^\circ$

$q=1$ $\theta_2 = \frac{3 \times 180^\circ}{4} = 135^\circ$

$q=2$ $\theta_3 = \frac{5 \times 180^\circ}{4} = 225^\circ$

$q=3$ $\theta_4 = \frac{7 \times 180^\circ}{4} = 315^\circ$

To find break away points:

$$C.E \Rightarrow 1 + G(s)H(s) = 0$$

$$\Rightarrow 1 + \frac{K}{s(s+4)(s^2+4s+10)} = 0$$

$$\Rightarrow s(s+4)(s^2+4s+10) + K = 0$$

$$\Rightarrow s^4 + 8s^3 + 36s^2 + 80s + K = 0$$

$$K = -[s^4 + 8s^3 + 36s^2 + 80s] \rightarrow \textcircled{1}$$

$$\frac{dK}{ds} = -[4s^3 + 24s^2 + 72s + 80] = 0$$

$$\Rightarrow s^3 + 6s^2 + 18s + 20 = 0$$

Roots are $s = -2, -2 \pm j2.45$

check for k

when $s = -2$

From $\textcircled{1} \Rightarrow K = -[s^4 + 8s^3 + 36s^2 + 80s]$
 $= 64 \rightarrow \text{positive and real}$

when $s = -2 \pm j2.45 = 3.16 \angle \pm 129^\circ$

$K = 100 \rightarrow \text{positive and real.}$

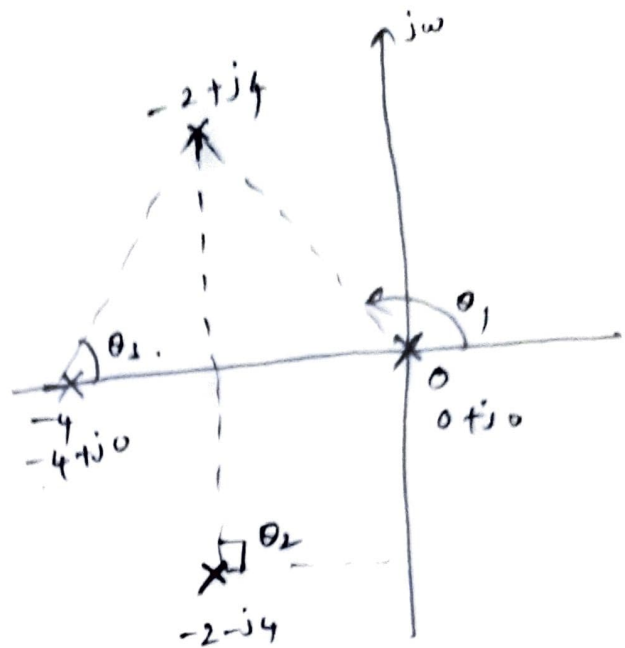
All roots are valid break away points.

To find angle of departure.

For $-2+j4$
 $\theta_2 = +90^\circ$

$$\begin{aligned}\theta_1 &= \tan^{-1} \left(\frac{0-4}{0+2} \right) \\ &= -\tan^{-1} 2 \\ &= 180^\circ - \tan^{-1} 2 \\ &= 117^\circ.\end{aligned}$$

$$\begin{aligned}\theta_3 &= \tan^{-1} \left(\frac{0-4}{-4+2} \right) \\ &= 63^\circ.\end{aligned}$$



$$\begin{aligned}\phi_d &= 180^\circ - [\theta_1 + \theta_2 + \theta_3 - 0] \\ &= 180^\circ - [117^\circ + 90^\circ + 63^\circ] \\ &= -90^\circ.\end{aligned}$$

For $-2-j4$

$\phi_d = +90^\circ$

Substituting point

$$s^4 + 8s^3 + 36s^2 + 80s + k = 0$$

s^4	1	36	k
s^3	8	80	10
s^2	26	k	
s^1	$\frac{260-k}{26}$		
s^0	k		

$$\frac{260-k}{26} = 0$$

$$k = 260$$

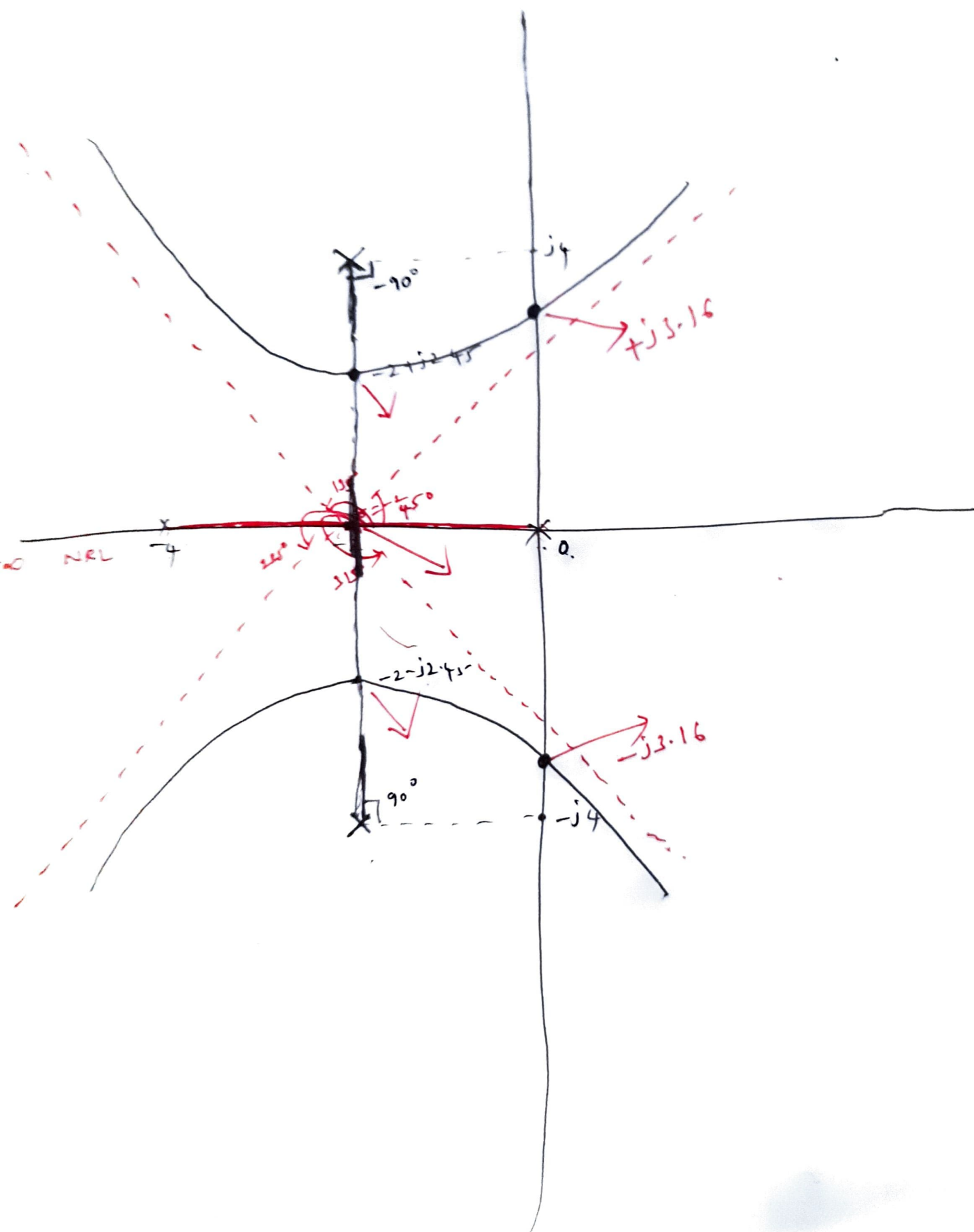
$$AE \Rightarrow 26s^2 + k = 0$$

$$26s^2 = -260$$

$$s^2 = -10$$

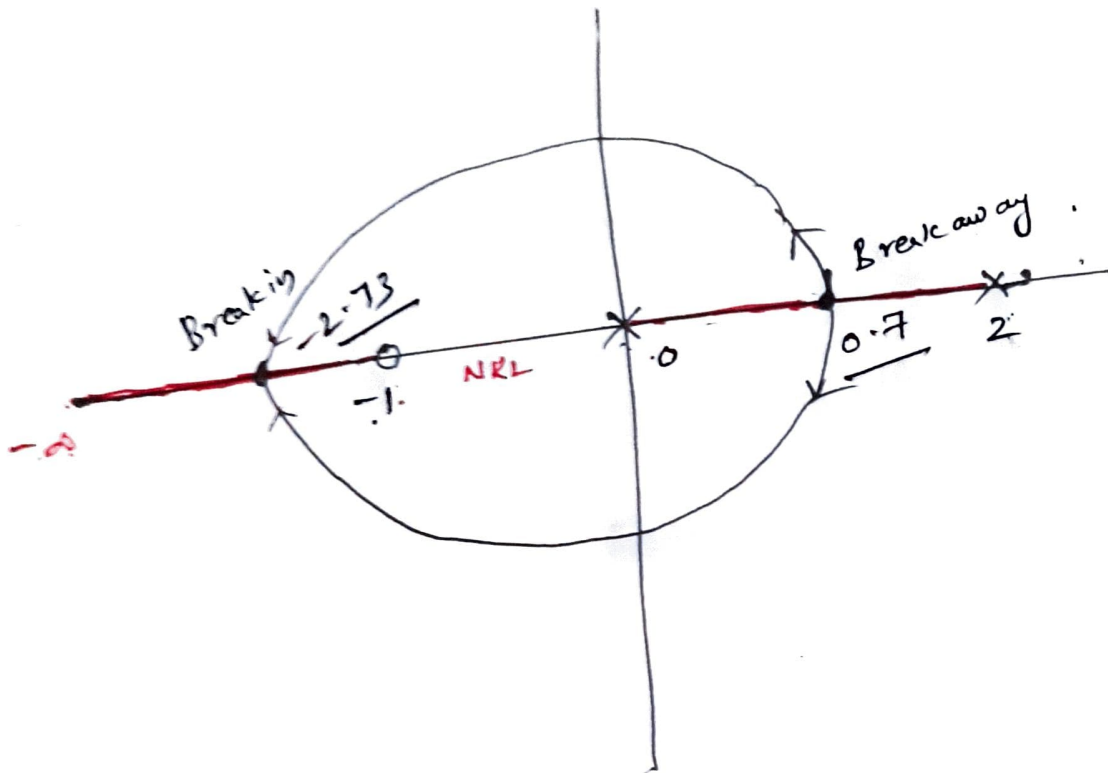
$$s = \sqrt{-10}$$

$$s = \pm j3.16$$



$$G(s) = \frac{K(s+1)}{s(s-2)}$$

Zero $\rightarrow s = -1$
 poles $s = 0, 2$



$$1 + G(s)H(s) = 0$$

$$1 + \frac{K(s+1)}{s(s-2)} = 0$$

$$s^2 - 2s + Ks + K = 0$$

$$K = -\frac{(s^2 - 2s)}{(s+1)}$$

$$\frac{dK}{ds} = \frac{(s+1)[2s-2] - (s^2-2s)}{(s+1)^2} = 0$$

$$2s^2 - 2s - 2 - s^2 + 2s = 0$$

$$s^2 - 2 = 0$$

$$s = -2.73, 0.7$$

Effect of addition of poles and zeros to $G(s)H(s)$ on Root locus :-

The addition of poles and zeros should be only in the left half of s -plane.

Addition of poles:-

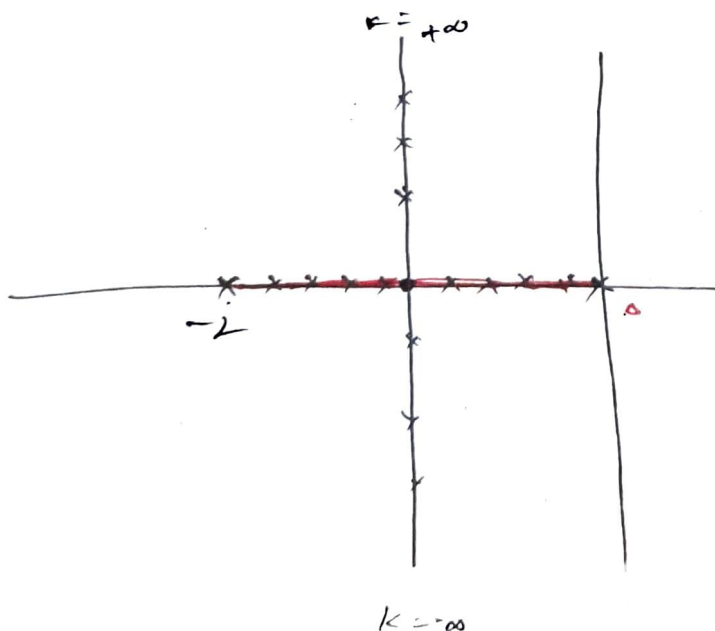
Let us consider $G(s)H(s) = \frac{k}{s}$

$$\frac{C(s)}{R(s)} = \frac{k}{s+k}$$



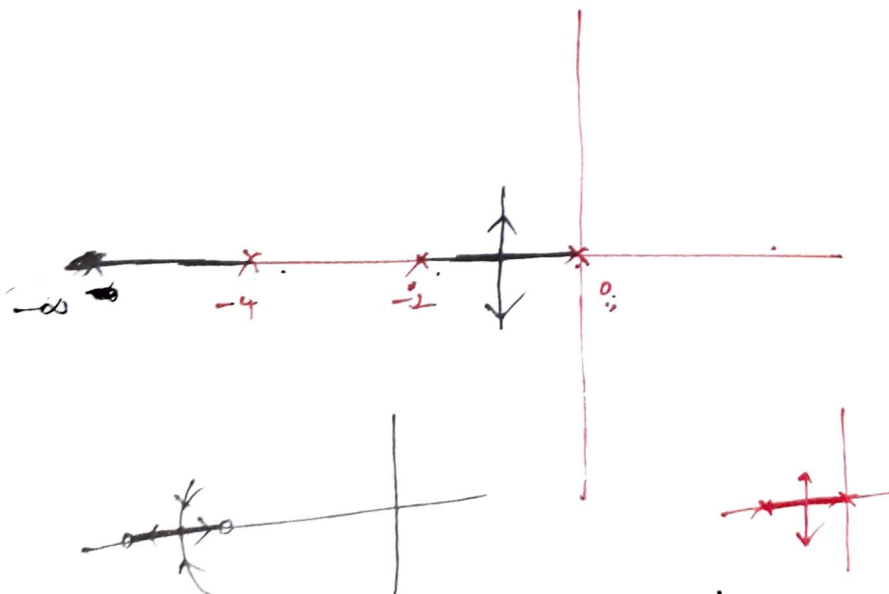
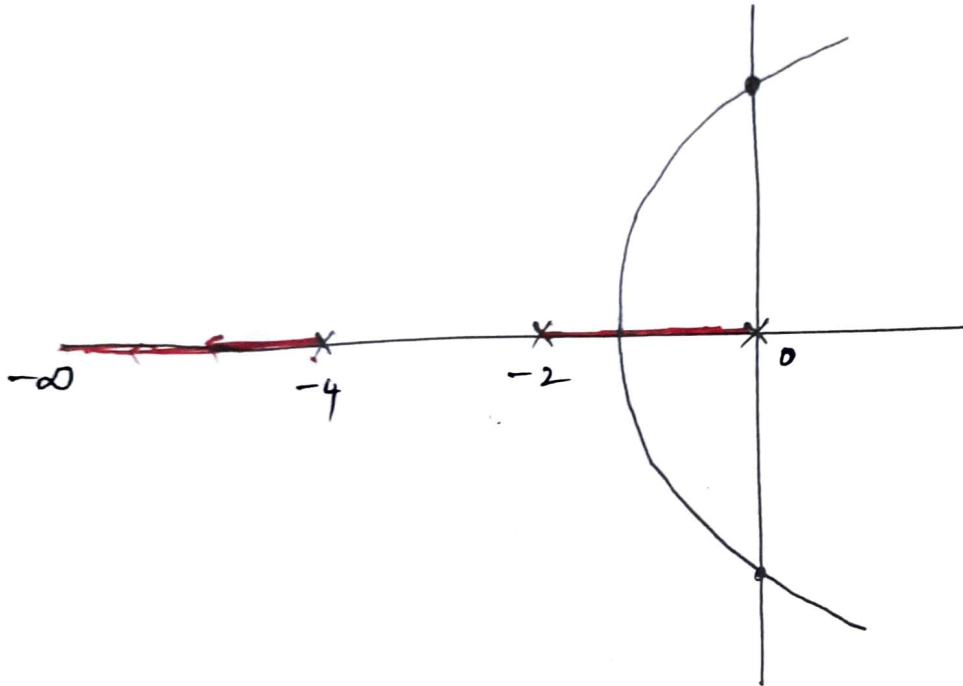
If we add a pole at $s=-2$ to the previous T.F.

$$G(s)H(s) = \frac{k}{s(s+2)}$$



If we add one more pole at $s = -4$ to $G(s)H(s)$

Then $G(s)H(s) = \frac{k}{s(s+2)(s+4)}$

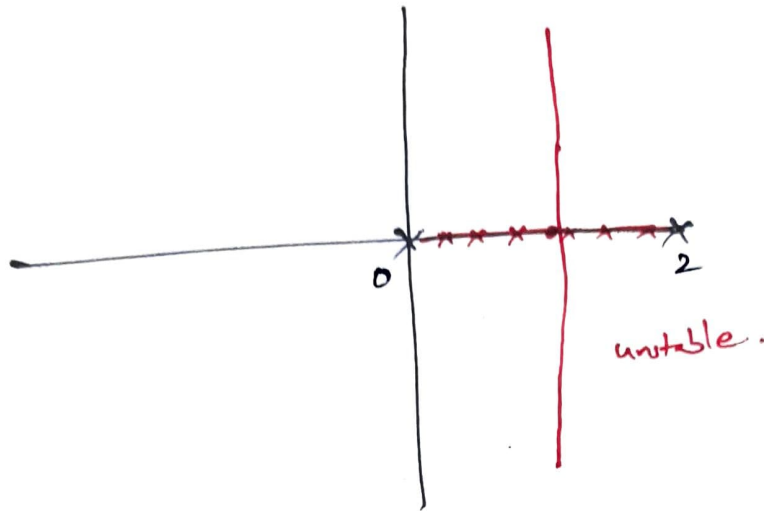


$$k = \infty$$

- 1) The root locus branches turn towards the RH of s-plane.
- 2) The break point shifted to right side.
- 3) The system becomes more oscillatory.
- 4) Relative stability decreases.
- 5) The Range of 'k' value for system stability decreases.
- 6) The damping factor ξ & ' ζ ' decreases.
- 7) As ' ζ ' decrease ' ω_d ' is ~~decreases~~ increases.
- 8) As ' ω_d ' increases time specifications t_d, t_r, t_p are decrease.
- 9) It shows that the transient performance is improved.
- 10) percentage of peak overshoot increases. As peak overshoot increases the system become less stable.
- 11) Time constant increases and hence settling time also increase.
- 12) BW is increases. As BW $\uparrow\uparrow$, the $t_r \downarrow\downarrow$.
- 13) The largest BW gives the very quick response.

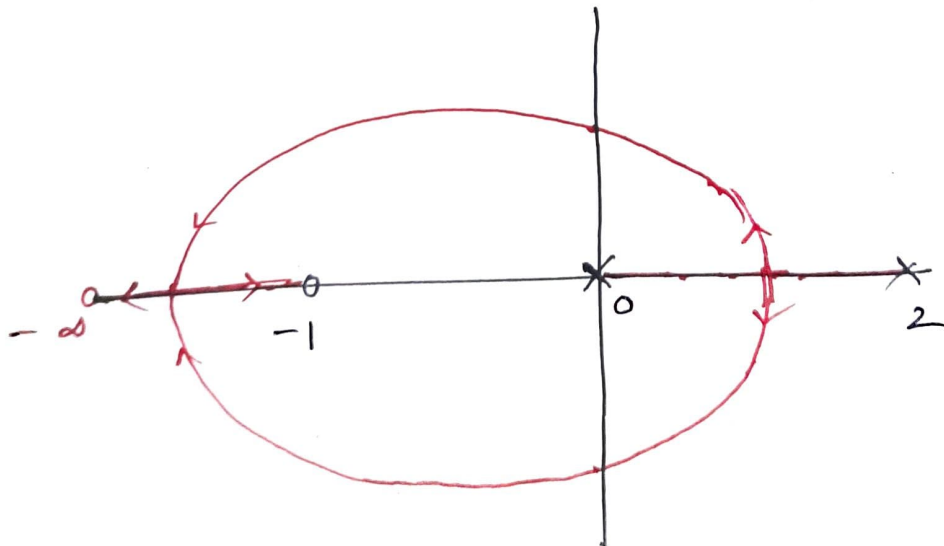
Addition of zeros:-

$$G(s)H(s) = \frac{K}{s(s-2)}$$



If we added one zero at $s=-1$ to the $G(s)H(s)$

$$G(s)H(s) = \frac{K(s+1)}{s(s-2)}$$



- 1) RL Branches turns left
- 2) BP shifted towards left
- 3) The system becomes less oscillatory
- 4) Relative stability increases
- 5) The Range of K' value increases for stability
- 6) ζ' increases
- 7) ω_d decreases, hence t_r, t_p, t_d increases
- 8) $\%M_p$ decreases, so the system more stable
- 9) settling time decreases, because time constant decreases
- 10) BW ~~decreases~~ because t_r increases.
- 11) The system response becomes low.

Note: If the zero added near to the origin then the result is similar to the effect of addition of poles. that means transient performance improved.

Frequency Response Analysis

Sinusoidal Transfer function:-

The response of the system for sinusoidal input is called sinusoidal response. The ratio of sinusoidal response and sinusoidal input is called sinusoidal Transfer function. It is denoted by $T(j\omega)$. It is also called as frequency domain T.F.

If the s-domain T.F, $T(s)$ is known then frequency domain T.F, $T(j\omega)$ can be obtained directly from $T(s)$ by replacing 's' by 'j\omega'

$$\begin{array}{ccc} \frac{r(t)}{A \sin(\omega t + \theta)} & \boxed{T(j\omega) = |T(j\omega)| \angle T(j\omega)} & \frac{c(t)}{B \sin(\omega t + \phi)} \\ \text{ALO} & & \text{where } B = A \times |T(j\omega)| \\ & & \phi = \theta + \angle T(j\omega) \end{array}$$

$$\begin{array}{ccc} 5 \cos(\omega t + 45^\circ) & \boxed{\frac{1}{1+j}} & c(t) = ? \\ |T(j\omega)| = \frac{1}{\sqrt{1^2+1^2}} = \frac{1}{\sqrt{2}} & & c(t) = \frac{5 \times 1}{\sqrt{2}} \cos(\omega t + 45^\circ - 45^\circ) \\ \angle T(j\omega) = -\tan^{-1}(1/1) = -45^\circ & & \boxed{c(t) = 5/\sqrt{2} \cos \omega t} \end{array}$$

$$\begin{array}{l} \frac{1}{a+jb} \quad M = \frac{1}{\sqrt{a^2+b^2}} \rightarrow \text{magnitude} \\ \phi = -\tan^{-1}(b/a) \rightarrow \text{phase} \end{array}$$

$$\begin{array}{l} a+jb \quad M = \sqrt{a^2+b^2} \\ \phi = \tan^{-1}(b/a) \end{array}$$

Frequency Response

The frequency domain T.F $T(j\omega)$ is a complex function of ω . Hence it can be separated as magnitude function and phase function. Now, the magnitude and phase functions will be real functions of ω and they are called frequency response

$$\begin{aligned}\text{open loop T.F: } G(s) &\xrightarrow{s=j\omega} G(j\omega) = |G(j\omega)| \angle G(j\omega) \\ \text{loop T.F } G(s)H(s) &\xrightarrow{s=j\omega} G(j\omega)H(j\omega) = |G(j\omega)H(j\omega)| \angle G(j\omega)H(j\omega) \\ \text{CLTF: } M(s) &\xrightarrow{s=j\omega} M(j\omega) = |M(j\omega)| \angle M(j\omega)\end{aligned}$$

Eg:

$$G(s) = \frac{1}{s(s+1)(s+2)}$$

$$G(j\omega) = \frac{1}{(j\omega+1)(j\omega+2)}$$

$$\begin{aligned}\frac{1}{a+jb} \quad M &= \frac{1}{\sqrt{a^2+b^2}} \\ \phi &= -\tan^{-1}(b/a)\end{aligned}$$

$$\begin{aligned}|G(j\omega)| &= \frac{1}{\sqrt{1+\omega^2} \sqrt{4+\omega^2}} \\ \angle G(j\omega) &= -\tan^{-1}(\omega/1) - \tan^{-1}(\omega/2)\end{aligned}$$

$$G(s) = \frac{1}{s(s+1)(s+2)}$$

$$G(j\omega) = \frac{1}{j\omega(j\omega+1)(j\omega+2)} \quad |G(j\omega)| = \frac{1}{\omega \sqrt{1+\omega^2} \sqrt{4+\omega^2}}$$

$$\begin{aligned}\angle G(j\omega) &= -\tan^{-1}(\omega/0) - \tan^{-1}(\omega/1) - \tan^{-1}(\omega/2) \\ &= -90^\circ - \tan^{-1}(\omega) - \tan^{-1}(\omega/2)\end{aligned}$$

Frequency Domain Specifications:-

- 1) Resonant peak (M_r)
- 2) Resonant frequency (ω_r)
- 3) Bandwidth (ω_b)
- 4) cut off rate
- 5) Gain Margin (K_g)
- 6) phase Margin (γ)

Resonant peak (M_r):-

The maximum value of the magnitude of the closed loop Transfer function is called Resonant peak (M_r).

Resonant frequency (ω_r):

The frequency at which resonant peak occurs is called resonant frequency (ω_r).

Bandwidth (ω_b):

The Bandwidth is range of frequencies in which the system normalized gain is more than -3 dB . The frequency at which the gain is -3 dB is called cutoff frequency.

Cutoff rate:-

The slope of log-magnitude ~~over~~ curve near the cut off frequency is called cutoff rate.

Gain Margin (K_g):-

The gain Margin is defined as the value of gain, to be added to the system, in order to bring the system to the verge of instability.

It is given by reciprocal of magnitude of OLTF at phase cross over frequency (ω_{pc}). The frequency at which the phase of OLTF is 180° is called ω_{pc} .

$$\text{Gain Margin } K_g = \frac{1}{|G(j\omega)|_{\omega=\omega_{pc}}}$$

$$K_g \text{ in db} = 20 \log \left| \frac{1}{G(j\omega_{pc})} \right|$$

Phase Margin (γ):

It is defined as the additional phase lag to be added at gain cross over frequency in order to bring the system to the verge of instability.

The gain cross over freq (ω_{gc}), is the frequency at which gain of OLTF is unity (or '0' in db).

$$\text{Phase Margin } \gamma = 180^\circ + \phi_{gc}$$

$$\text{where } \phi_{gc} = \angle G(j\omega) \Big|_{\omega=\omega_{gc}}$$

Frequency domain specifications of second order system:

Resonant peak (M_r)

Consider the CLTF of second order system.

$$\frac{C(s)}{R(s)} = M(s) = \frac{\omega_n^2}{s^2 + 2\xi\omega_n s + \omega_n^2}$$

The sinusoidal T.F. $M(j\omega)$ is obtained by putting $s = j\omega$.

$$\begin{aligned} M(j\omega) &= \frac{\omega_n^2}{(j\omega)^2 + 2\xi\omega_n(j\omega) + \omega_n^2} \\ &= \frac{\omega_n^2}{-\omega^2 + j2\xi\omega_n\omega + \omega_n^2} = \frac{\omega_n^2}{\omega_n^2 \left[\frac{-\omega^2}{\omega_n^2} + j2\xi\frac{\omega}{\omega_n} + 1 \right]} \\ &= \frac{1}{1 - \left[\frac{\omega}{\omega_n} \right]^2 + j2\xi\frac{\omega}{\omega_n}} \end{aligned}$$

Let us assume, normalized frequency $u = \frac{\omega}{\omega_n}$

$$\therefore M(j\omega) = \frac{1}{1 - u^2 + j2\xi u}$$

$$\begin{aligned} \text{Magnitude } M &= |M(j\omega)| = \frac{1}{\sqrt{(1-u^2)^2 + (2\xi u)^2}} \\ M &= [(1-u^2)^2 + 4\xi^2 u^2]^{-1/2} \rightarrow \textcircled{1} \end{aligned}$$

$$\text{phase angle } \alpha = \angle M(j\omega) = -\tan^{-1} \left[\frac{2\xi u}{1-u^2} \right]$$

The resonant peak is max. value of M . The condition for max. value of M can be obtained by differentiating the equation of M w.r.t u and letting $\frac{dM}{du} = 0$ when $u = u_r$ where $u_r = \frac{\omega_r}{\omega_n} = \text{Normalized Resonant freq.}$

on differentiating equation ① w.r.t. 'u' we get

$$\begin{aligned}\frac{dm}{du} &= \frac{d}{du} \left[(1-u^2)^2 + 4\xi^2 u^2 \right]^{-1/2} \\ &= -\frac{1}{2} \left[(1-u^2)^2 + 4\xi^2 u^2 \right]^{-3/2} [2(1-u^2)(-2u) + 8\xi^2 u] \\ &= \frac{-4u(1-u^2) + 8\xi^2 u}{2 \left[(1-u^2)^2 + 4\xi^2 u^2 \right]^{3/2}} \rightarrow \textcircled{2}\end{aligned}$$

Replace u by u_r in ② and equate to zero

$$\begin{aligned}\frac{4u_r(1-u_r^2) - 8\xi^2 u_r}{2 \left[(1-u_r^2)^2 + 4\xi^2 u_r^2 \right]^{3/2}} &= 0 \\ \Rightarrow 4u_r(1-u_r^2) - 8\xi^2 u_r &= 0 \Rightarrow 4u_r^3 = 4u_r - 8\xi^2 u_r \\ \Rightarrow u_r^2 &= 1 - 2\xi^2 \Rightarrow \boxed{u_r = \sqrt{1-2\xi^2}}\end{aligned}$$

put this condition in equation of 'm'

$$\begin{aligned}m_r &= \left. \frac{1}{\left[(1-u^2)^2 + 4\xi^2 u^2 \right]^{1/2}} \right|_{u=u_r} = \frac{1}{\left[(1-u_r^2)^2 + 4\xi^2 u_r^2 \right]^{1/2}} \\ &= \frac{1}{\left[(1-(1-2\xi^2))^2 + 4\xi^2 (1-2\xi^2) \right]^{1/2}} = \frac{1}{\left[4\xi^4 + 4\xi^2 - 8\xi^4 \right]^{1/2}} \\ &= \frac{1}{\left[4\xi^2 - 4\xi^4 \right]^{1/2}} = \frac{1}{2\xi \sqrt{1-\xi^2}}\end{aligned}$$

Resonant peak $\boxed{m_r = \frac{1}{2\xi \sqrt{1-\xi^2}}}$

Resonant frequency (ω_r):

$$\text{Normalized resonant frequency } u_r = \frac{\omega_r}{\omega_n} \\ = \sqrt{1 - 2\xi^2}$$

$$\text{The resonant frequency } \boxed{\omega_r = \omega_n \sqrt{1 - 2\xi^2}}$$

Band width (ω_b)

$$\text{Let, Normalized Bandwidth } u_b = \frac{\omega_b}{\omega_n}$$

when $u = u_b$, the magnitude $|m|$ is $\frac{1}{\sqrt{2}}$ (0 - 3 dB)

Hence in the equation for $|m|$, put $u = u_b$ and equate to $\frac{1}{\sqrt{2}}$

$$\therefore M = \frac{1}{[(1 - u_b^2)^2 + 4\xi^2 u_b^2]^{1/2}} = \frac{1}{\sqrt{2}}$$

$$\Rightarrow (1 - u_b^2)^2 + 4\xi^2 u_b^2 = 2$$

$$\Rightarrow 1 + u_b^4 - 2u_b^2 + 4\xi^2 u_b^2 = 2$$

$$\Rightarrow u_b^4 - 2u_b^2(1 - 2\xi^2) - 1 = 0$$

$$\text{Let } x = u_b^2, \quad \therefore x^2 - 2x(1 - 2\xi^2) - 1 = 0$$

$$\therefore x = \frac{2(1 - 2\xi^2) \pm \sqrt{4(1 - 2\xi^2)^2 + 4}}{2}$$

$$= \frac{2(1 - 2\xi^2) \pm 2\sqrt{1 + 4\xi^4 - 4\xi^2 + 1}}{2}$$

$$\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Let us take only the positive sign.

$$\therefore x = 1 - 2\xi^2 + \sqrt{2 - 4\xi^2 + 4\xi^4}$$

But $u_b = \sqrt{x}$

$$\therefore u_b = \sqrt{1 - 2\xi^2 + \sqrt{2 - 4\xi^2 + 4\xi^4}}$$

But we have $u_b = \frac{\omega_b}{\omega_n}$

Bandwidth $\omega_b = \omega_n \cdot u_b$

$$\boxed{\omega_b = \omega_n \left[1 - 2\xi^2 + \sqrt{2 - 4\xi^2 + 4\xi^4} \right]^{\frac{1}{2}}}$$

Resonant peak $M_r = \frac{1}{2\xi\sqrt{1-\xi^2}}$

ξ, ω_n

Resonant freq $\omega_r = \omega_n \sqrt{1 - 2\xi^2}$

Bandwidth $\omega_b = \omega_n \left[\sqrt{1 - 2\xi^2} + \sqrt{2 - 4\xi^2 + 4\xi^4} \right]$

Frequency Response plots:-

The various graphical methods available for frequency response analysis are.

- 1) Bode plot ✓
 - 2) polar plot / Nyquist plot ✓
 - 3) Nichols plot
 - 4) Mand N circles
 - 5) Nichols chart
- drawn for OLTF to find CL system stability
- drawn for CLTF

Bode plot:

It is a frequency response plot of sinusoidal T.F.

Bode plot consists of two graphs.

one is plot of magnitude versus $\log \omega \rightarrow$ Magnitude plot

second is plot of phase angle versus $\log \omega \rightarrow$ phase plot.

Ex. Consider the OLTF, $G(s) = \frac{K(1+sT_1)}{s(1+sT_2)(1+sT_3)}$

$$G(j\omega) = \frac{K(1+j\omega T_1)}{j\omega(1+j\omega T_2)(1+j\omega T_3)}$$

$$\frac{K+j0}{\sqrt{K^2} = K}$$

$$\text{Magnitude } |G(j\omega)| = \frac{K \sqrt{1+\omega^2 T_1^2}}{\omega \sqrt{1+\omega^2 T_2^2} \sqrt{1+\omega^2 T_3^2}}$$

$$\text{Magnitude in dB} = 20 \log |G(j\omega)|$$

$$= 20 \log \left[\frac{K \sqrt{1+\omega^2 T_1^2}}{\omega \sqrt{1+\omega^2 T_2^2} \sqrt{1+\omega^2 T_3^2}} \right]$$

$$|G(j\omega)| \text{ in dB} = 20 \log K + 20 \log \sqrt{1 + \omega^2 T_1^2} - 20 \log \omega - 20 \log \sqrt{1 + \omega^2 T_2^2} - 20 \log \sqrt{1 + \omega^2 T_3^2}$$

$$\text{phase angle of } G(j\omega) = \angle G(j\omega) = +\tan^{-1}\left(\frac{\omega T_1}{1}\right) - \tan^{-1}\left(\frac{\omega T_2}{1}\right) - \tan^{-1}\left(\frac{\omega T_3}{1}\right)$$

$$\angle G(j\omega) = -\tan^{-1}(\omega T_1) - 90^\circ - \tan^{-1}(\omega T_2) - \tan^{-1}(\omega T_3)$$



Basic factors of $G(j\omega)$:

- 1) Constant gain, 'K'
- 2) Integral factor $\frac{K}{j\omega}$ (d) $\frac{K}{(j\omega)^n}$
- 3) Derivative factor $K \times j\omega$ (d) $K \times (j\omega)^n$
- 4) First order factor in denominator $\frac{1}{1+j\omega T}$ (d) $\frac{1}{(1+j\omega T)^n}$
- 5) First " " " numerator $(1+j\omega T)$ (d) $(1+j\omega T)^n$

Constant gain K:-

$$\text{Let } G(s) = K$$

$$\therefore G(j\omega) = K$$

$$M = |G(j\omega)| = \sqrt{K^2} = K$$

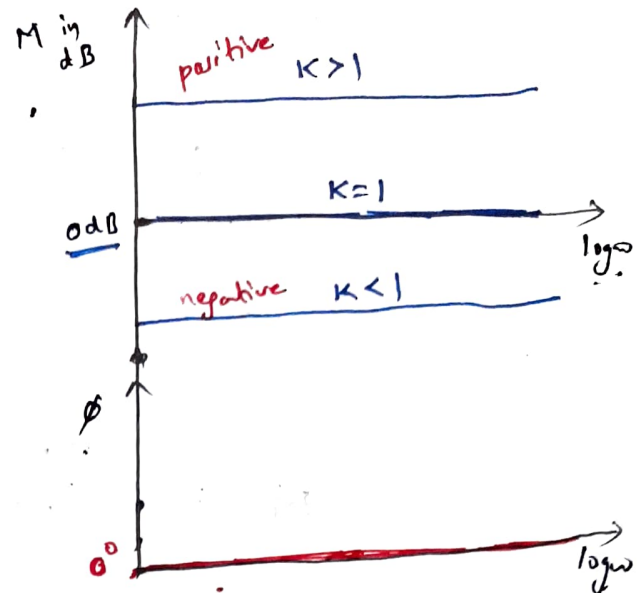
$$\phi = \angle G(j\omega) = \tan^{-1}\left\{\frac{0}{K}\right\} = \underline{0^\circ}$$

$$M \text{ in dB} = 20 \log K$$

$$\text{when } K=1 \Rightarrow M = 0 \text{ dB (zero)}$$

$$K > 1 \Rightarrow M = \text{positive}$$

$$K < 1 \Rightarrow M = \text{negative}$$



Integral factor:-

$$\text{Let } G(s) = \frac{K}{s}$$

$$\therefore G(j\omega) = \frac{K}{j\omega} = -j \frac{K}{\omega}$$

$$M = \sqrt{0^2 + \left(\frac{K}{\omega}\right)^2} = \frac{K}{\omega}$$

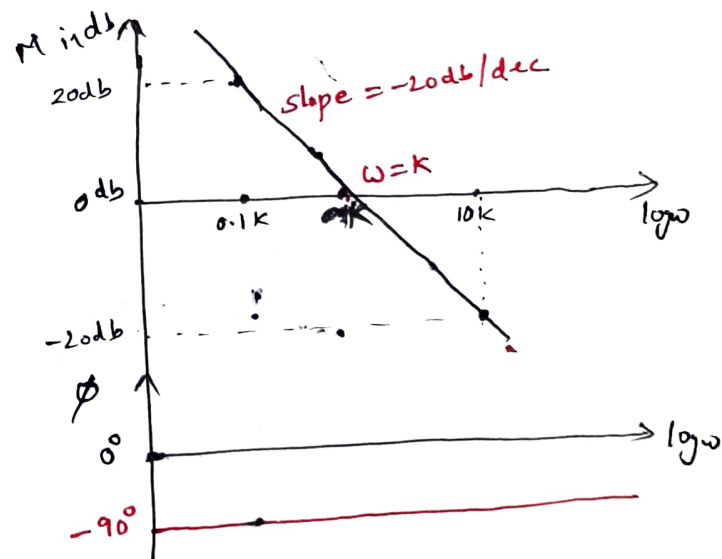
$$M \text{ in dB} = 20 \log \frac{K}{\omega}$$

$$\phi = -\tan^{-1}\left\{\frac{K/\omega}{0}\right\} = -90^\circ$$

$$\text{when } \omega = 0.1K, \quad M = 20 \log \left(\frac{K}{0.1K}\right) = 20 \log 10 = 20 \text{ dB}$$

$$\omega = K, \quad M = 20 \log \left(\frac{K}{K}\right) = 20 \log 1 = 0 \text{ dB}$$

$$\omega = 10K, \quad M = 20 \log \left(\frac{K}{10K}\right) = 20 \log 0.1 = -20 \text{ dB}$$



When an integral factor has multiple poles, then

$$G(s) = \frac{K}{s^n}$$

$$G(j\omega) = \frac{K}{(j\omega)^n}$$

$$M = \frac{K}{\omega^n}$$

$$M \text{ in dB} = 20 \log \left(\frac{K}{\omega^n} \right) = 20 \log \left(\frac{K^{1/n}}{\omega} \right)^n$$

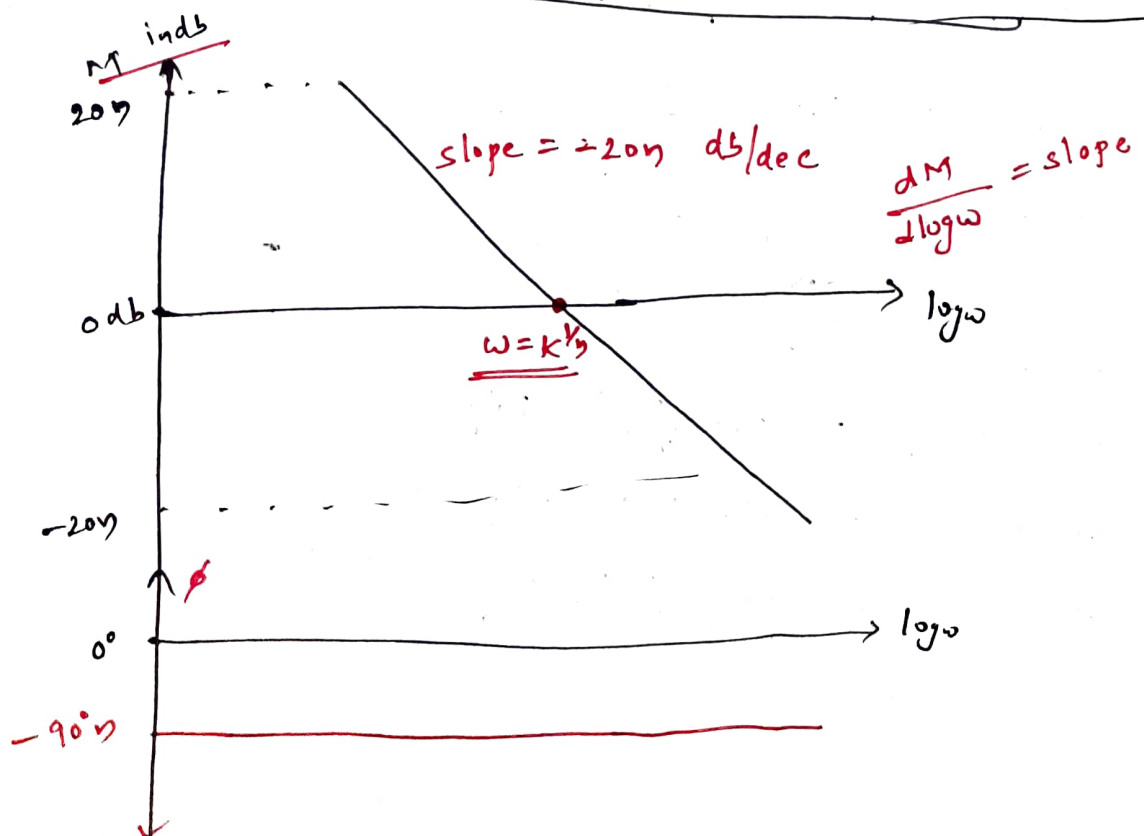
$$= \underline{20n \log \left(\frac{K^{1/n}}{\omega} \right)}$$

$$\phi = n \tan^{-1} \left(\frac{0\omega}{0} \right) = -90^\circ n$$

$$\frac{1}{s^3} = \frac{1}{(j\omega)(j\omega)(j\omega)}$$

$$= -\tan^{-1} \left(\frac{\omega}{0} \right) - \tan^{-1} \left(\frac{\omega}{0} \right) - \tan^{-1} \left(\frac{\omega}{0} \right)$$

$$= -90 - 90 - 90 = -270^\circ$$



Derivative factor 1-

$$\text{Let } G(s) = Ks$$

$$\therefore G(j\omega) = K \cdot j\omega$$

$$M = \sqrt{(K\omega)^2} = K\omega$$

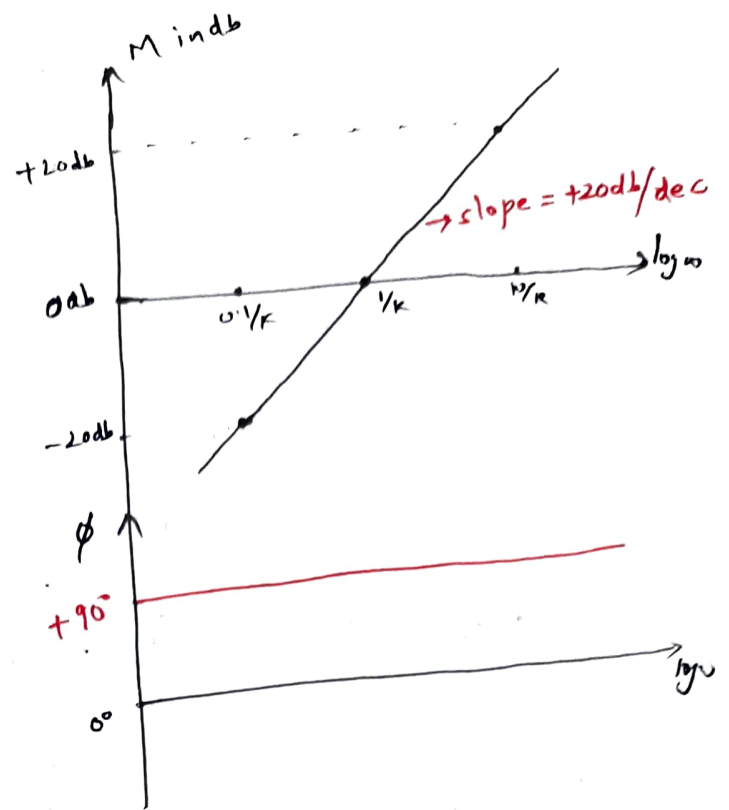
$$M_{\text{indb}} = 20 \log(K\omega)$$

$$\phi = +\tan^{-1}\left[\frac{K\omega}{0}\right] = +90^\circ$$

$$\text{when } \omega = 0.1/K, \quad M = -20\text{db}$$

$$\omega = 1/K, \quad M = 0\text{db}$$

$$\omega = 10/K, \quad M = +20\text{db}$$



when derivative factor has multiple ^{zeros} poles, then

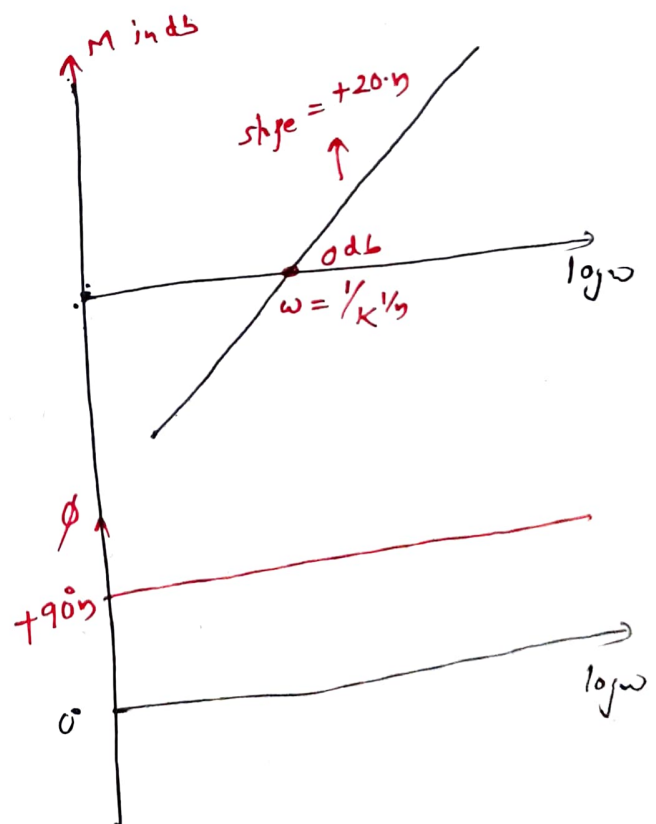
$$G(s) = K \cdot s^n$$

$$\therefore G(j\omega) = K(j\omega)^n$$

$$M = K(\omega)^n$$

$$M_{\text{indb}} = 20 \log(K\omega^n) \\ = \underline{+20n \log[K^n \cdot \omega]}$$

$$\phi = n \tan^{-1}\left[\frac{K\omega}{0}\right] = \underline{+90^\circ n}$$



First order factor in denominator:

$$G(s) = \frac{1}{1 + sT}$$

$$G(j\omega) = \frac{1}{1 + j\omega T}$$

$$M = \frac{1}{\sqrt{1 + \omega^2 T^2}}, \quad \phi = -\tan^{-1}(\omega T)$$

$$M_{\text{in db}} = 20 \log \left[\frac{1}{\sqrt{1 + \omega^2 T^2}} \right] = -20 \log \sqrt{1 + \omega^2 T^2}$$

At very low frequencies, $\omega T \ll 1$, $M = -20 \log \sqrt{1 + \omega^2 T^2}$
 $\omega \ll 1/T$, $= -20 \log 1 = \underline{0 \text{ db}}$

At high frequencies, $\omega T \gg 1$, $M = -20 \log \sqrt{1 + \omega^2 T^2}$
 $\omega \gg 1/T$, $= \underline{-20 \log(\omega T)}$

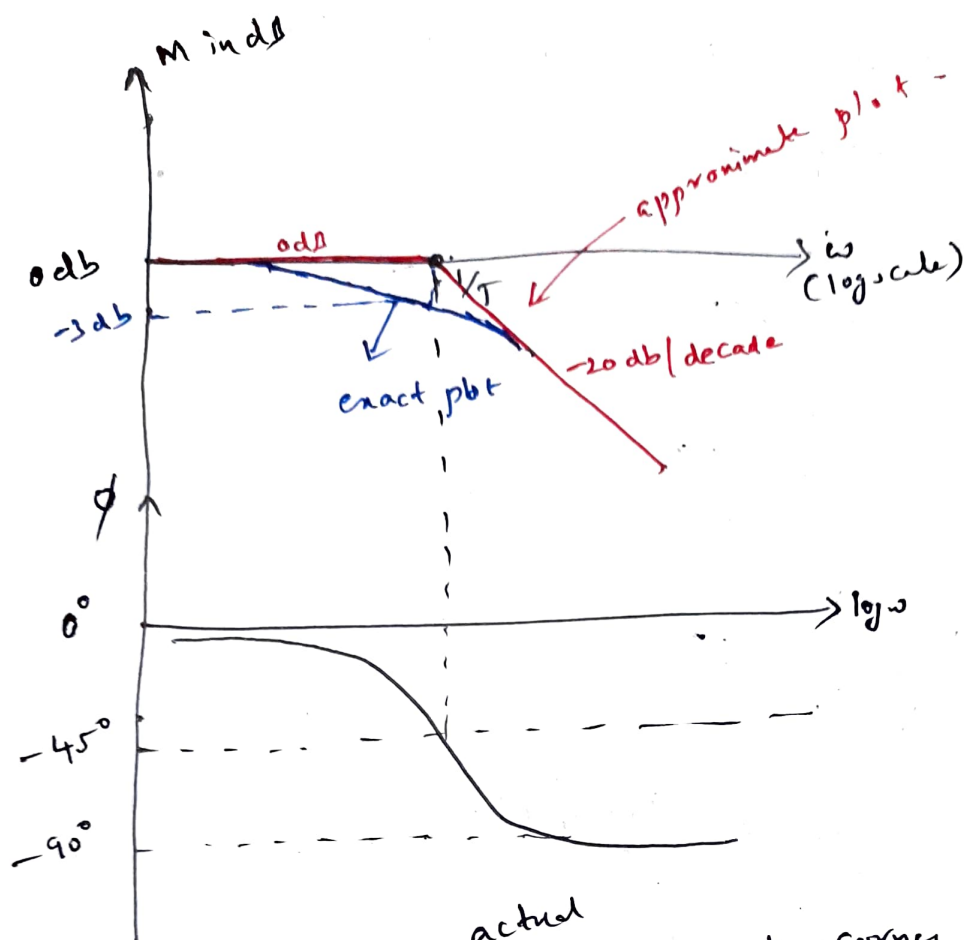
when $\omega = 1/T$, $M = -20 \log 1 = 0 \text{ db}$

$\omega = 10/T$, $M = -20 \log 10 = -20 \text{ db}$

From the above analysis, the factor $1/(1 + j\omega T)$ can be approximated by two straight lines. one is straight line at 0 db in the frequency range $0 < \omega < 1/T$ and the other line with slope -20 db/dec in the frequency range $1/T < \omega < \infty$.

The frequency at which the two asymptotes meet is called corner frequency, or break frequency.

i.e. $\omega_c = 1/T$



But the ^{actual} magnitude at corner frequency $\omega_c = 1/\tau$ is

$$M = -20 \log \sqrt{1^2 + 1} = -20 \log \sqrt{2} \\ = -3 \text{ dB}$$

phase angle $\phi = \angle G(j\omega) = -\tan^{-1} \omega \tau$

At corner freq. $\omega = \omega_c = 1/\tau$, $\phi = -\tan^{-1} 1 \\ = -45^\circ$

As $\omega \rightarrow 0$, $\phi \rightarrow 0^\circ$
 $\omega \rightarrow \infty$, $\phi \rightarrow -90^\circ$

when the first order factor in denominator has a multiplicity of 'm', then

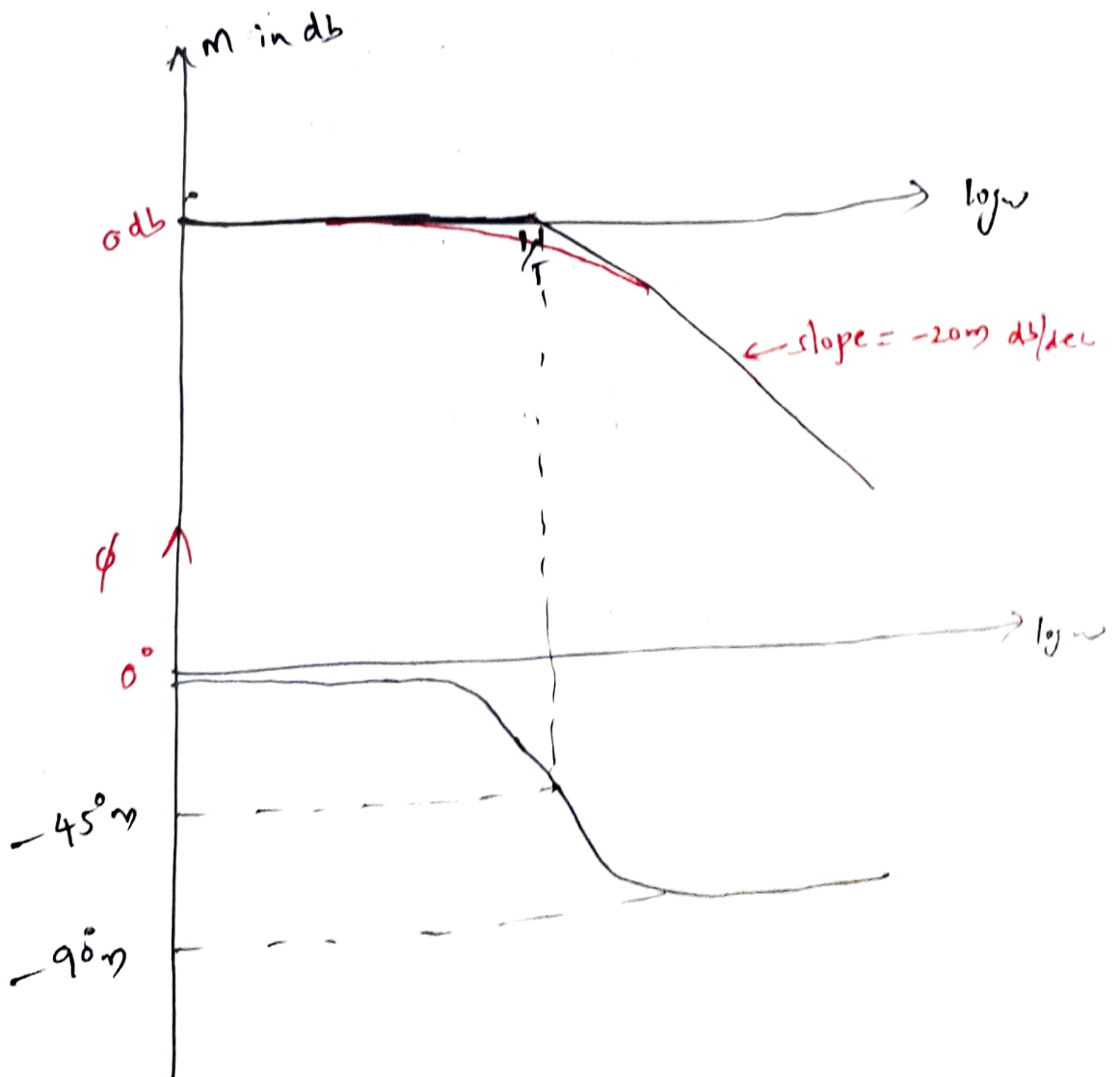
$$G(s) = \frac{1}{(1+sT)^m}$$

$$G(j\omega) = \frac{1}{(1+j\omega T)^m}$$

$$M = \frac{1}{\left(\sqrt{1+\omega^2 T^2}\right)^m}$$

$$M_{\text{in db}} = -20m \log \sqrt{1+\omega^2 T^2}$$

$$\phi = -m \tan^{-1} \omega T$$



First order factor in numerator:

$$G(s) = 1 + sT$$

$$G(j\omega) = 1 + j\omega T$$

$$M = \sqrt{1 + \omega^2 T^2}$$

$$M_{in\ db} = 20 \log \sqrt{1 + \omega^2 T^2}$$

$$\phi = \tan^{-1}(\omega T)$$

$$\omega \ll 1/T \Rightarrow M = 0\ db$$

$$\omega \gg 1/T \Rightarrow M = +20\ db/dec.$$

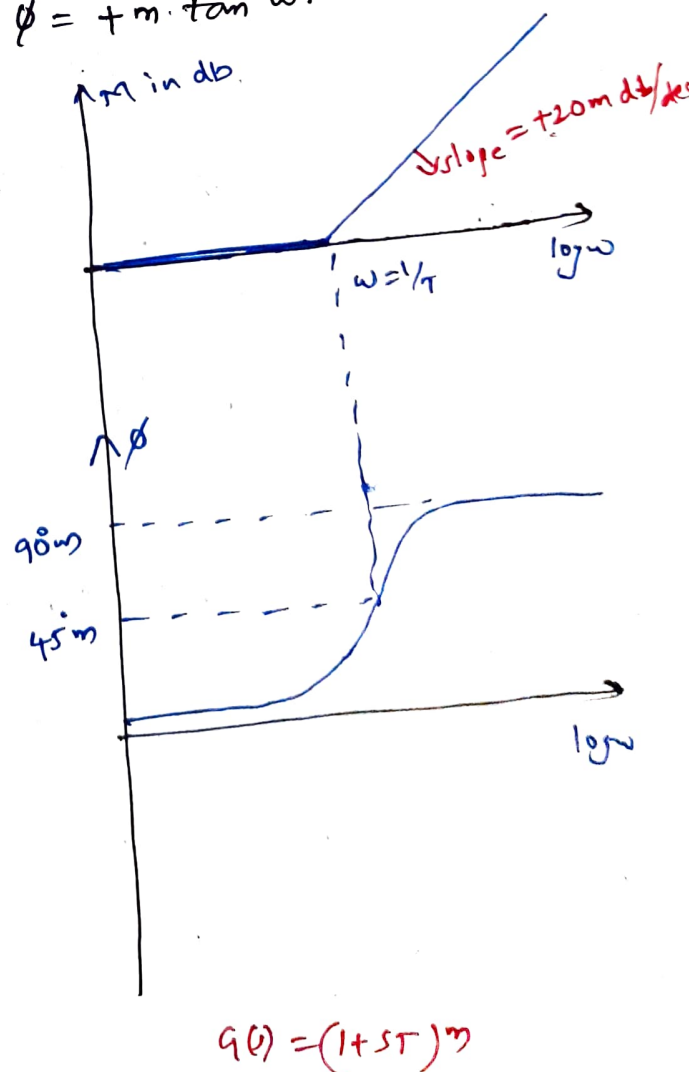
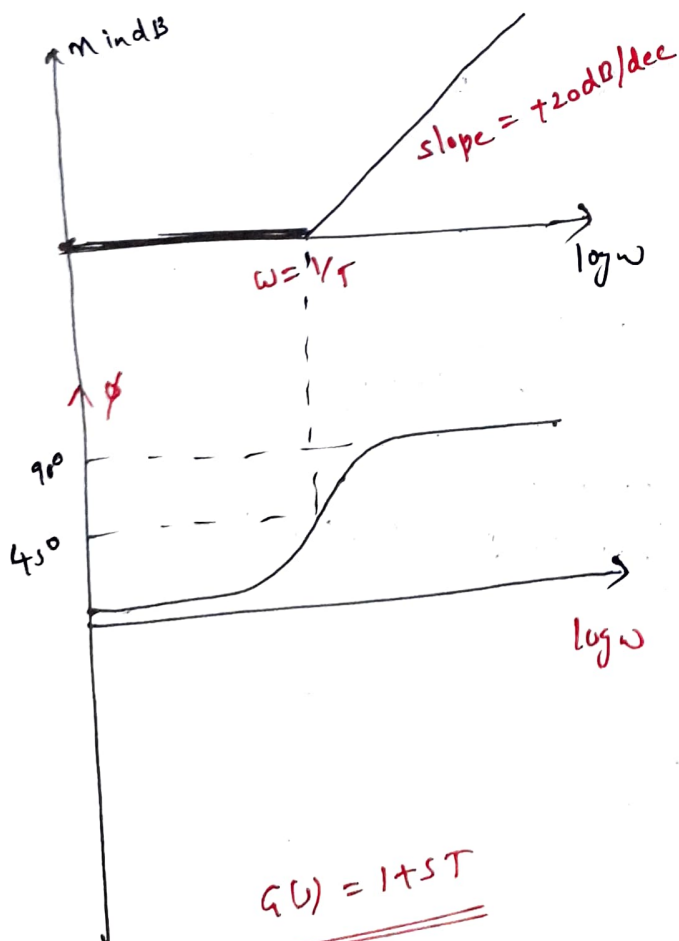
If $G(s) = (1 + sT)^m$

$$G(j\omega) = (1 + j\omega T)^m$$

$$M = (\sqrt{1 + \omega^2 T^2})^m$$

$$M_{in\ db} = 20m \log \sqrt{1 + \omega^2 T^2}$$

$$\phi = +m \cdot \tan^{-1} \omega T$$



procedure in magnitude plots.

From the above analysis, the following conclusions can be obtained.

- i) The constant gain K , integral, and derivative factors contribute gains (magnitude) at all frequencies.
- ii) In approximate plot the first, quadratic and higher order factors contribute gain only when $\text{freq} > \text{corner freq}$

Hence, low frequency response up to the lowest corner frequency is decided by K or $\frac{K}{(\omega)^n}$ or $K(\omega)^n$

$\therefore$ The magnitude plot can be started with K or $\frac{K}{(\omega)^n}$ or $K(\omega)^n$ and then db magnitude of every first and higher order terms are added one by one in the increasing order of corner frequency.

Eg.

$$\text{let } G(s) = \frac{K(1+sT_1)^2}{s^2(1+sT_2)(1+sT_3)}$$

$$G(\omega) = \frac{K(1+j\omega T_1)^2}{(\omega)^2(1+j\omega T_2)(1+j\omega T_3)}$$

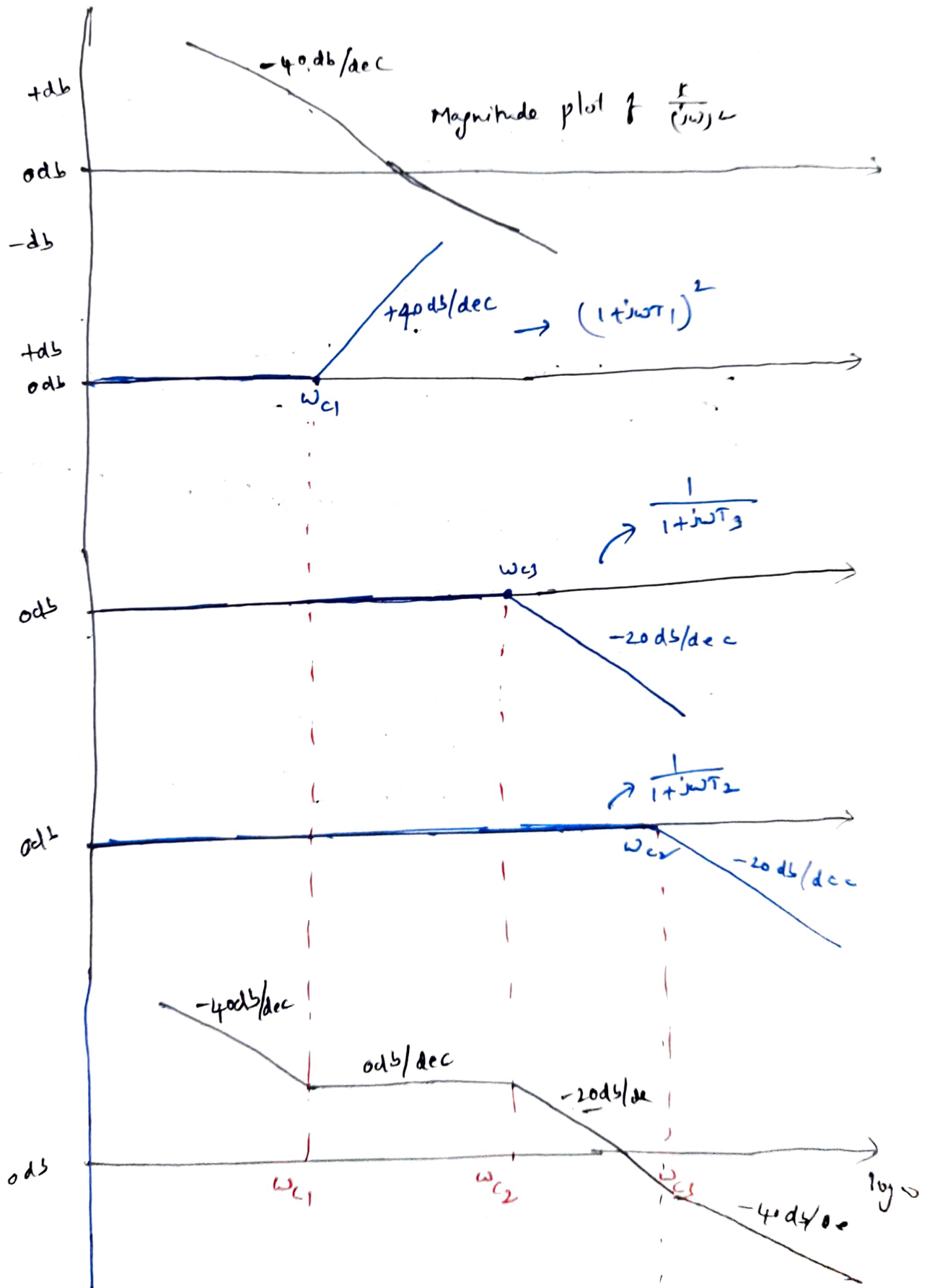
$$\text{Let } T_2 < T_3 < T_1$$

The corner freq are $\omega_{c1} = 1/T_1$, $\omega_{c2} = 1/T_2$, $\omega_{c3} = 1/T_3$

$$\text{then } \omega_{c1} < \omega_{c3} < \omega_{c2}$$

$$G(s) = \frac{K \cdot (1 + j\omega T_1)^2}{(j\omega)^2 (1 + j\omega T_2) (1 + j\omega T_3)}$$

Let $T_2 < T_1 < T_3$ then $\omega_{c1} < \omega_{c3} < \omega_{c2}$



The step by step procedure for plotting the magnitude plot:-

Step 1: convert the given T.F into Bode form & time constant form. The Bode form of T.F is

$$G(s) = \frac{K(1+sT_1)}{s(1+sT_2)(1+sT_3)} \xrightarrow{s=j\omega} G(j\omega) = \frac{K(1+j\omega T_1)}{j\omega(1+j\omega T_2)(1+j\omega T_3)}$$

Step 2: List the corner frequencies in increasing order and prepare a table as shown below.

Term	Corner frequency rad/sec	slope db/dec	change in slope db/dec

In the above table enter K (or) $\frac{K}{j\omega}$ (or) $K(j\omega)$ as the first term and other terms in the increasing order of corner frequencies.

Step 3: choose an arbitrary frequency ω_x which is lesser than the lowest corner frequency. calculate the db magnitude of $'K'$ (or) $\frac{K}{j\omega}$ (or) $K(j\omega)$ at ω_x and at the low corner frequency.

Step 4: Then calculate the gain at every corner frequency one by one by using the below formula.

$$\begin{aligned} \text{Gain at } \omega_y &= \text{change in slope from } \omega_x \text{ to } \omega_y + \text{Gain at } \omega_x \\ &= \left(\text{slope from } \omega_x \text{ to } \omega_y \right) \times \log \frac{\omega_y}{\omega_x} + \text{Gain at } \omega_x \\ &\quad \downarrow \text{change in slope} \end{aligned}$$

step 5: Choose an arbitrary frequency ' ω_h ' which is higher than highest corner frequency. Calculate gain at ' ω_h ' by using formula in step '4'.

step 6: In a semilog graph sheet mark all the required range of freq. on x-axis (log scale) and the range of db magnitude on y-axis (ordinary scale).

step 7: Mark all the points obtained in steps (3), (4) and (5) on the graph and joint the points by straight line and mark slope at every part of the graph.

↓ approximate plot.

procedure for phase plot of bode plot.

The phase plot is an exact plot and no approximation is made. Hence the exact phase angles of $G(j\omega)$ are computed for various values of ' ω '. (preferably the freq. chosen for magnitude plot).

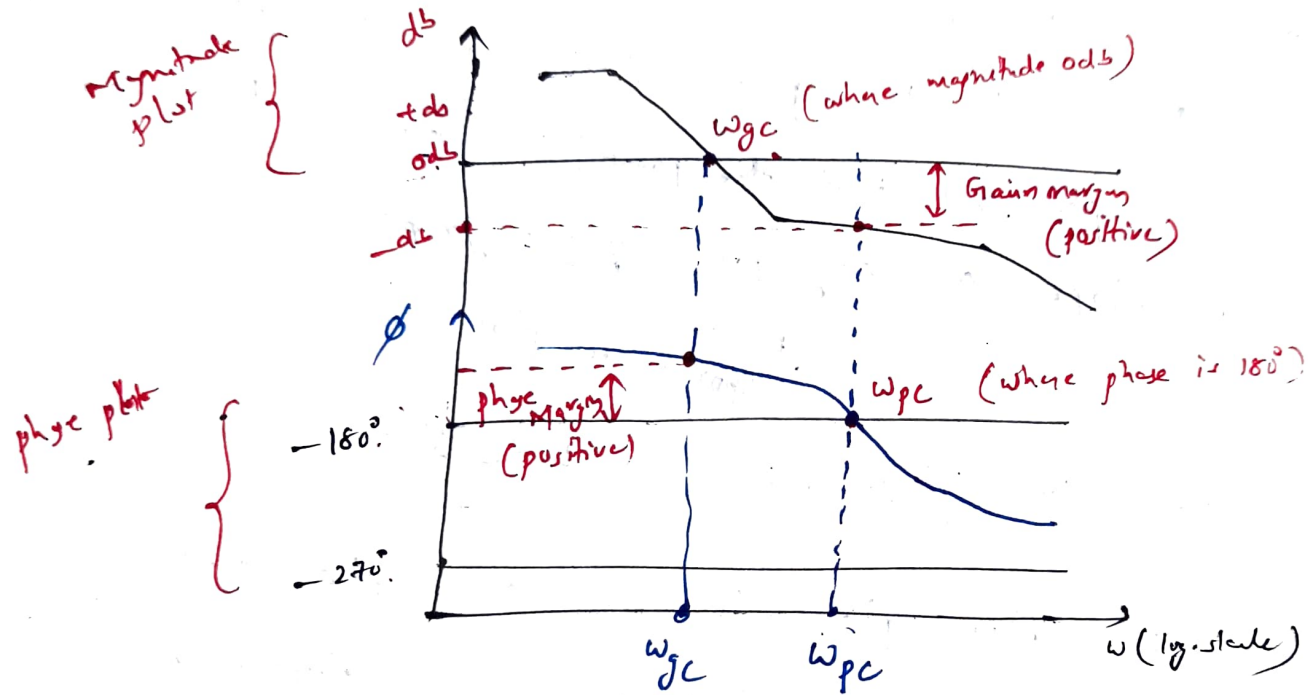
Determination of gain margin and phase margin from bode plots

The gain margin in db is given by the negative of db magnitude of $G(j\omega)$ at ω_{pc} . The ω_{pc} is the freq. at which phase of $G(j\omega)$ is -180° .
For stable system gain margin should be positive in db.

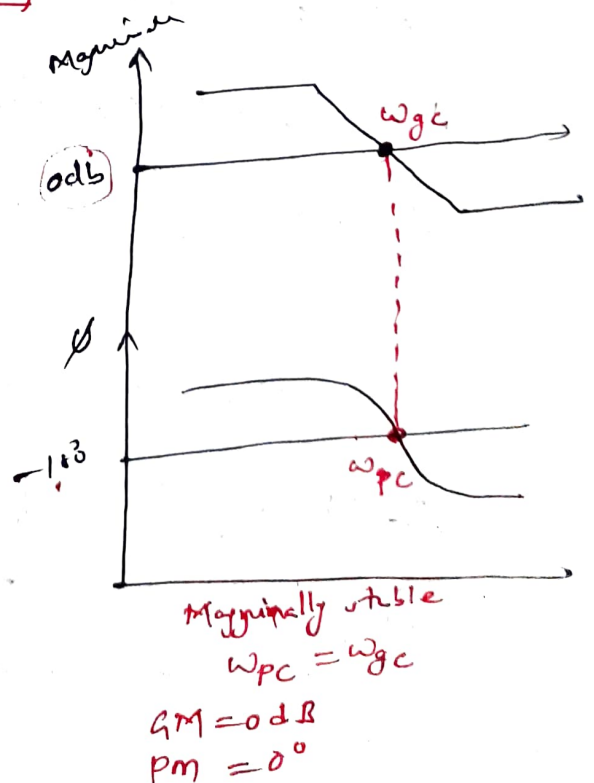
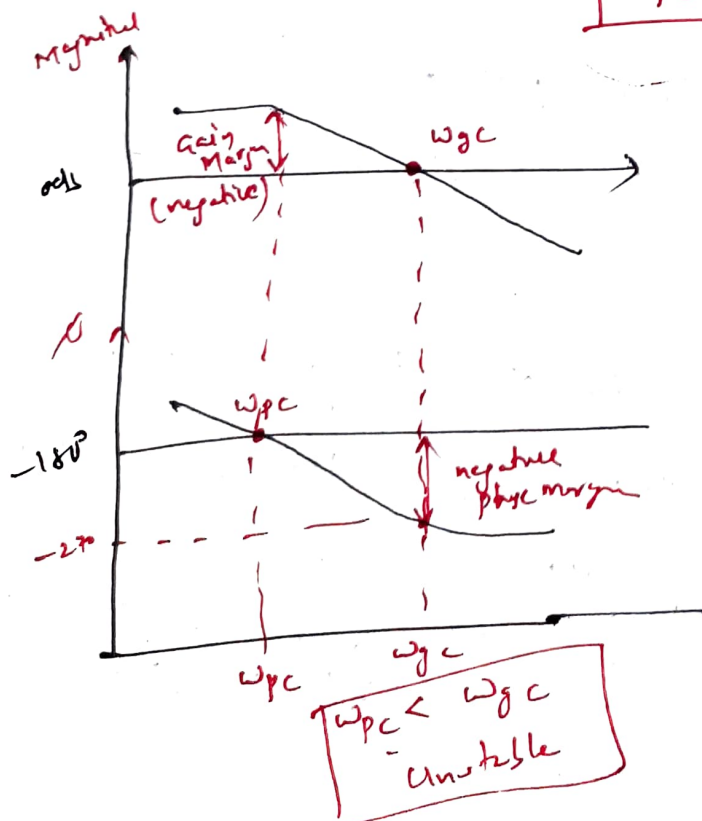
Let ϕ_{gc} is the phase angle of $G(s)$ at gain cross over frequency (ω_{gc}). The ω_{gc} is the frequency at which the db magnitude of $G(s)$ is zero. Now

phase margin $\gamma = 180^\circ + \phi_{gc}$

For stable system phase margin should be positive.



$\boxed{\omega_{pc} > \omega_{gc}}$ for stable system



P.0

Sketch Bode plot for the following T.F and determine the system gain 'K' for the gain cross over frequency to be 5 rad/sec.

$$G(s) = \frac{Ks^2}{(1+0.2s)(1+0.02s)}$$

Sol:

The sinusoidal T.F $G(j\omega)$ is obtained by replacing 's' with 'j ω ' in the given T.F.

$$G(j\omega) = \frac{K(j\omega)^2}{(1+0.2j\omega)(1+0.02j\omega)}$$

Let $K=1$

$$G(j\omega) = \frac{(j\omega)^2}{(1+0.2j\omega)(1+0.02j\omega)}$$

Magnitude plot:

$$\frac{|G(j\omega)|}{|G(s)|} = \frac{1}{1+5T} \quad \omega_c = \frac{1}{T} \text{ rad/sec}$$

The corner frequencies are $\omega_{c1} = \frac{1}{0.2} = 5 \text{ rad/sec}$

$$\omega_{c2} = \frac{1}{0.02} = 50 \text{ rad/sec}$$

Term	corner frequency rad/sec	slope dB/dec	change in slope dB/dec
$(j\omega)^2$	-	+40	-
$\frac{1}{1+0.2j\omega}$	$\omega_{c1} = \frac{1}{0.2} = 5$	-20	$40-20 = 20$
$\frac{1}{1+0.02j\omega}$	$\omega_{c2} = \frac{1}{0.02} = 50$	-20	$20-20 = 0 \text{ dB}$

Let $\omega_l = 0.5 \text{ rad/sec}$ and $\omega_h = 100 \text{ rad/sec}$

Choose a low frequency such that $\omega_l < \omega_{c1}$ and choose

high frequency ω_h such that $\omega_h > \omega_{c2}$.

Let $M = |G(j\omega)|$ in dB

Let us calculate 'M' at $\omega_l, \omega_{c1}, \omega_{c2}, \omega_h$.

$$\begin{aligned} \text{At } \omega = \omega_l, \quad M &= 20 \log |(j\omega)^2| = 20 \log \omega^2 \\ &= 20 \log (0.5)^2 = -12 \text{ dB} \end{aligned}$$

$$\begin{aligned} \text{At } \omega = \omega_{c1}, \quad M &= 20 \log |(j\omega)^2| = 20 \log \omega^2 \\ &= 20 \log (5)^2 = +28 \text{ dB} \end{aligned}$$

$$\begin{aligned} \text{At } \omega = \omega_{c2}, \quad M &= \left[\text{slope from } \omega_{c1} \text{ to } \omega_{c2} \times \log \frac{\omega_{c2}}{\omega_{c1}} \right] + M \Big|_{\omega = \omega_{c1}} \\ &= +20 \times \log \frac{50}{5} + 28 = 48 \text{ dB} \end{aligned}$$

$$\begin{aligned} \text{At } \omega = \omega_h, \quad M &= \left[\text{slope from } \omega_{c2} \text{ to } \omega_h \times \log \frac{\omega_h}{\omega_{c2}} \right] + M \Big|_{\omega = \omega_{c2}} \\ &= 0 \text{ dB} \times \log \frac{100}{50} + 48 = 48 \text{ dB} \end{aligned}$$

Phase plot:

phase angle of $G(j\omega)$ is given by

$$G(j\omega) = \frac{K(j\omega)^2}{(1+0.2j\omega)(1+0.02j\omega)}$$

$$\begin{aligned} \phi = \angle G(j\omega) &= \tan^{-1}\left(\frac{j\omega}{0}\right) + \tan^{-1}\left(\frac{j\omega}{0}\right) - \tan^{-1}\left(\frac{0.2\omega}{1}\right) - \tan^{-1}\left(\frac{0.02\omega}{1}\right) \\ \phi &= 180^\circ - \tan^{-1}(0.2\omega) - \tan^{-1}(0.02\omega) \end{aligned}$$

ω rad/sec	$\theta_1 =$ $\tan^{-1}(0.2\omega)$ (deg)	$\theta_2 =$ $\tan^{-1}(0.02\omega)$ (deg)	$\phi = 180^\circ - \theta_1 - \theta_2$
0.5	5.7	0.6	$173.7 \approx 174$
0.1	11.2	1.1	168
5	45	5.7	130
10	63.4	11.2	106
50	84.3	45	50
100	87.1	63.4	30

Calculation of k :

Given that

$$\omega_{gc} = 5 \text{ rad/sec.}$$

↓

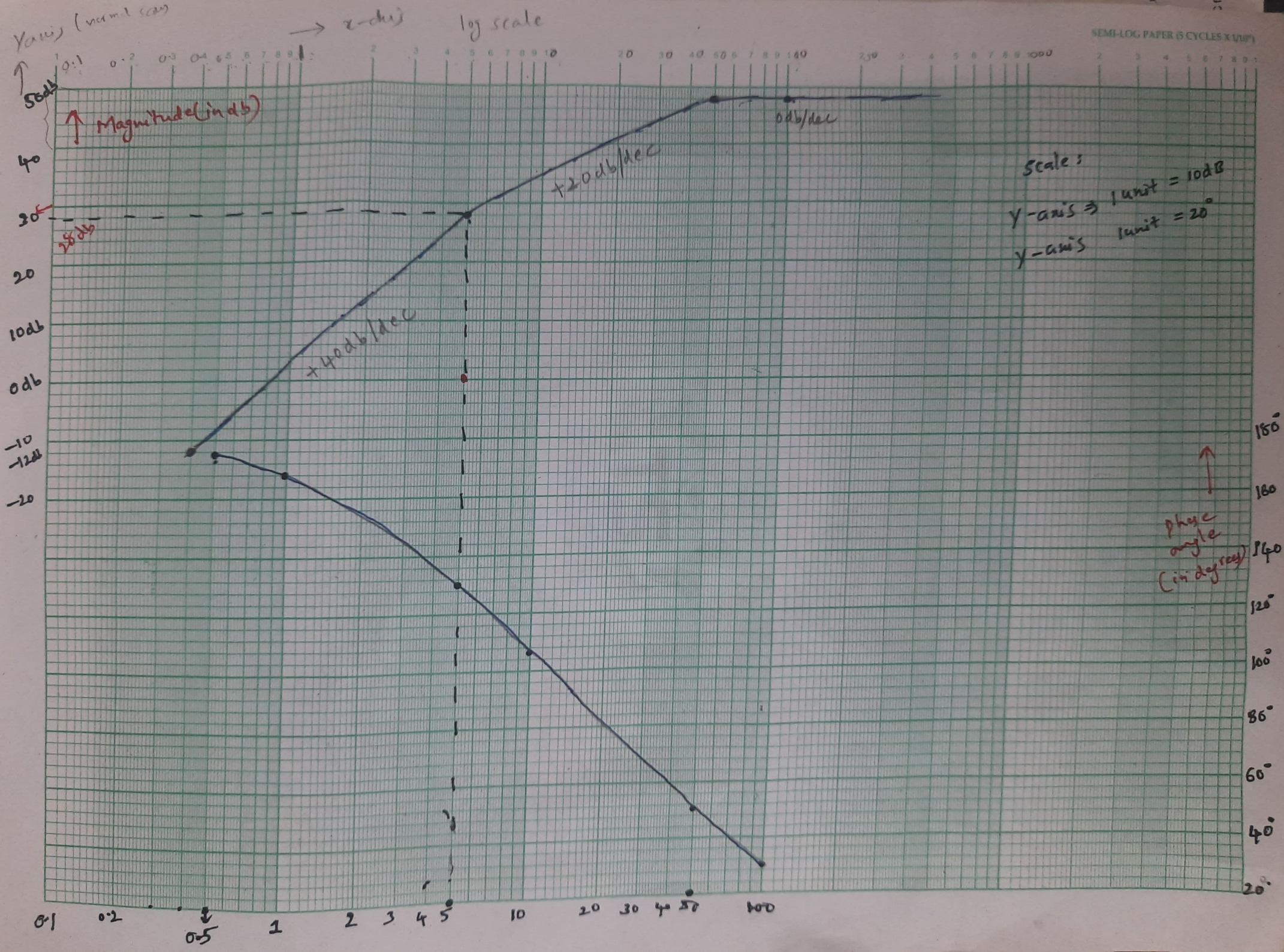
It is the frequency at which gain = 0 dB

The value of k is calculated by equating $20 \log k$ to -28 dB .

$$\therefore 20 \log_{10} k = -28 \text{ dB}$$

$$\log_{10} k = \frac{-28}{20} = -1.4$$

$$k = 10^{-28/20} = 0.0398$$



(P2). plot the bode diagram for the following OLTF and obtain gain and phase crossover frequencies

$$G(s) = \frac{10}{s(1+0.4s)(1+0.1s)}$$

Sol's The sinusoidal T-F of $G(j\omega)$ is obtained by replacing 's' by 'j\omega'.

$$G(j\omega) = \frac{10}{j\omega(1+0.4j\omega)(1+0.1j\omega)}$$

Magnitude plot:

The corner frequencies are

$$\omega_{c1} = \frac{1}{0.4} = 2.5 \text{ rad/sec}$$

$$\omega_{c2} = \frac{1}{0.1} = 10 \text{ rad/sec}$$

Term	corner freq. rad/sec	slope db/dec	change in slope db/dec
$\frac{10}{j\omega}$	—	-20	—
$\frac{1}{1+0.4j\omega}$	$\omega_{c1} = \frac{1}{0.4} = 2.5$	-20	$-20 - 20 = -40$
$\frac{1}{1+0.1j\omega}$	$\omega_{c2} = \frac{1}{0.1} = 10$	-20	$-40 - 20 = -60$

Let $\omega_l = 0.1 \text{ rad/sec}$, $\omega_h = 100 \text{ rad/sec}$

Let $M = |G(j\omega)|$ in db

Let us calculate M' at $\omega_1, \omega_{c1}, \omega_{c2}, \omega_h$

$$\text{At } \omega = \omega_1, \quad M = 20 \log \left| \frac{10}{j\omega} \right| = 20 \log \left(\frac{10}{\omega} \right) \\ = 20 \log \left(\frac{10}{0.1} \right) = +40 \text{ dB.}$$

$$\omega = \omega_{c1}, \quad M = 20 \log \left| \frac{10}{j\omega} \right| = 20 \log \left(\frac{10}{\omega} \right) \\ = 20 \log \left| \frac{10}{2.5} \right| = +12 \text{ dB}$$

$$\text{At } \omega = \omega_{c2}, \quad M = \left[\text{slope from } \omega_{c1} \text{ to } \omega_{c2} \times \log \frac{\omega_{c2}}{\omega_{c1}} \right] + M \Big|_{\omega = \omega_{c1}} \\ = -40 \times \log \frac{10}{2.5} + 12 = -12 \text{ dB}$$

$$\text{At } \omega = \omega_h, \quad M = \left[\text{slope from } \omega_{c2} \text{ to } \omega_h \times \log \frac{\omega_h}{\omega_{c2}} \right] + M \Big|_{\omega = \omega_{c2}} \\ = -60 \times \log \frac{100}{10} - 12 = -72 \text{ dB.}$$

Phase plot:

The phase angle of $G(s)$ is given by

$$\phi = -90^\circ - \tan^{-1}(0.4\omega) - \tan^{-1}(0.1\omega)$$

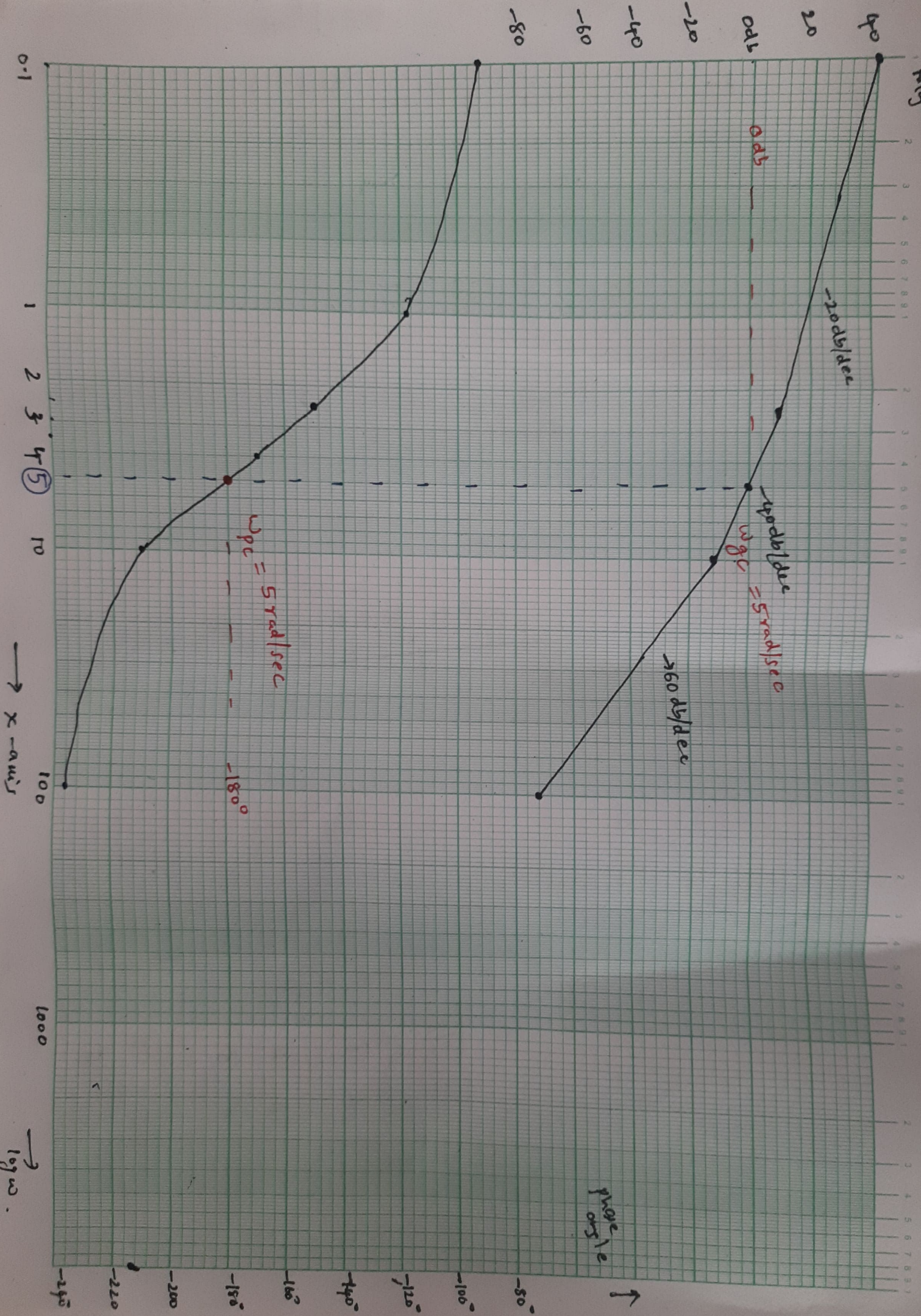
$$\begin{aligned} a+js &\Rightarrow \theta = \tan^{-1}(b/a) \\ \frac{1}{a+js} &\Rightarrow \theta = -\tan^{-1}(b/a) \\ s &\rightarrow j\omega \Rightarrow \theta = \tan^{-1}(\omega/0) = 90^\circ \\ \frac{1}{s} &\rightarrow \frac{1}{j\omega} \Rightarrow \theta = -90^\circ \\ s^2 &\rightarrow +180^\circ \\ \frac{1}{s^2} &\rightarrow -180^\circ \\ s^3 &\rightarrow +270^\circ \\ \frac{1}{s^3} &\rightarrow -270^\circ \end{aligned}$$

$$\begin{aligned} \frac{10}{j\omega(1+0.4s)(1+0.1s)} \\ s \rightarrow j\omega \quad \tan^{-1}\left(\frac{\omega}{0}\right) = 90^\circ \\ \frac{1}{s} \rightarrow \frac{1}{j\omega} \quad -\tan^{-1}\left(\frac{\omega}{0}\right) = -90^\circ \end{aligned}$$

ω	$\theta_1 = \tan^{-1}(0.4\omega)$	$\theta_2 = \tan^{-1}(0.1\omega)$	$\phi = -90^\circ - \theta_1 - \theta_2$
			-92
0.1	2.29°	0.51	
1	21.80°	5.71	-118
2.5	45.	14.0	-150
4	57.99	21.8	-170
10	75.96	45.0	-210
20	82.87	63.43	-236

Y-axis
Magnitude (in db)

SEMI-LOG PAPER (5 CYCLES X 100)



P3

$$G(s) = \frac{20}{s(1+3s)(1+4s)}$$

obtain ω_{gc} .

$$G(j\omega) = \frac{20}{j\omega(1+3j\omega)(1+4j\omega)}$$

$$\omega_{c1} = \frac{1}{4} = 0.25 \text{ rad/sec} \quad \omega_{c2} = \frac{1}{3} = 0.33 \text{ rad/sec}$$

Term	Coring freq.	slope	change in slope
$\frac{20}{\omega}$	-	-20	-
$\frac{1}{1+4j\omega}$	$\omega_{c1} = \frac{1}{4} = 0.25$	-20	$-20 - 20 = -40$
$\frac{1}{1+3j\omega}$	$\omega_{c2} = \frac{1}{3} = 0.33$	-20	$-40 - 20 = -60$

$$\omega_L = 0.1, \quad \omega_H = 1 \text{ rad/sec}$$

At ω_L $M = 20 \log \left| \frac{20}{j\omega} \right| = 20 \log \left| \frac{20}{0.1} \right| = 46 \text{ dB}$

At ω_{c1} $M = 20 \log \left| \frac{20}{\omega} \right| = 20 \log \left| \frac{20}{0.25} \right| = 38 \text{ dB}$

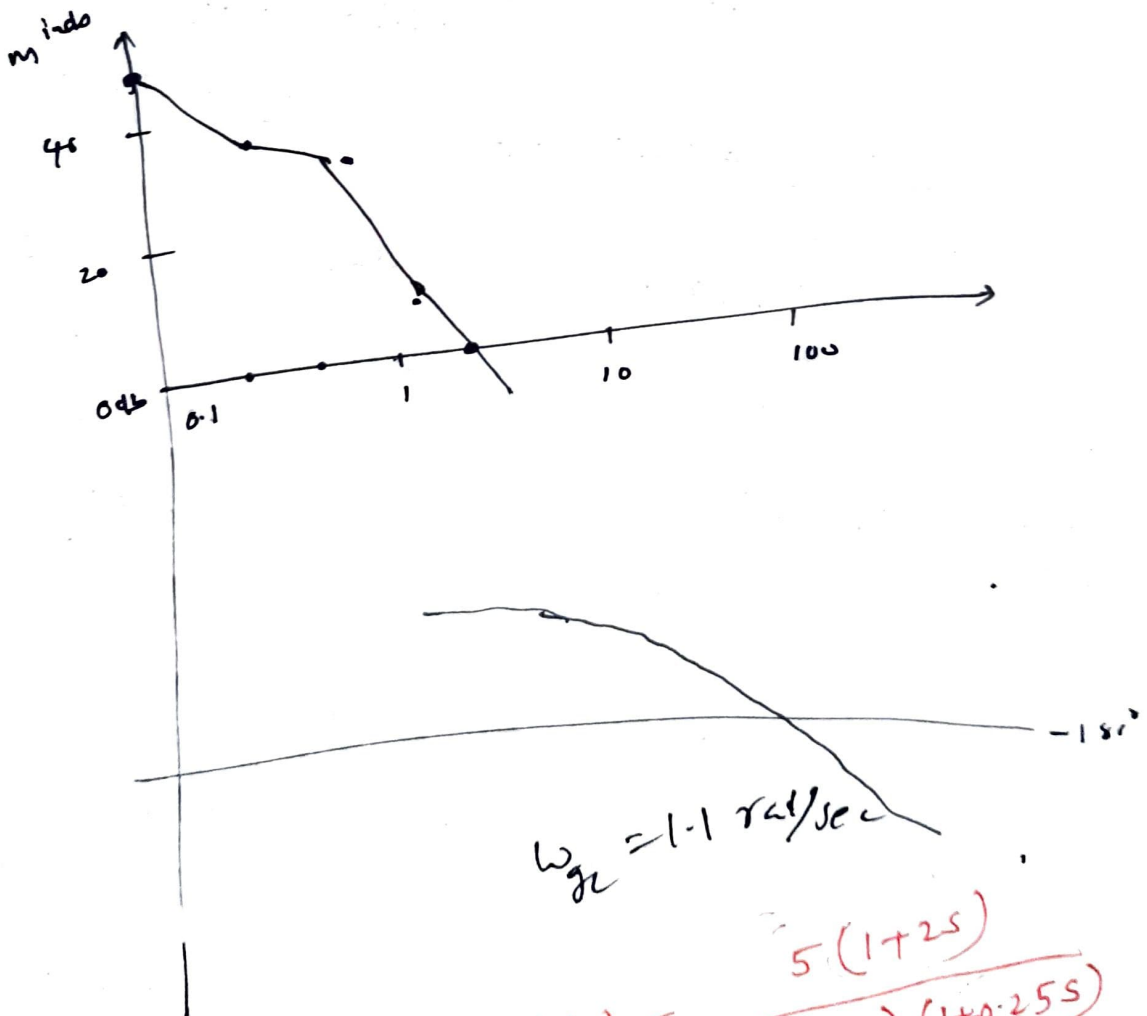
ω_{c2} $M = -40 \times \log \frac{0.33}{0.25} + 38 = 33 \text{ dB}$

At ω_H $M = -60 \times \log \frac{1}{0.33} + 33 = 4 \text{ dB}$

phase angle

$$\phi = -90^\circ - \tan^{-1} 3\omega - \tan^{-1} 4\omega$$

ω	$\tan^{-1} 3\omega$	$\tan^{-1} 4\omega$	ϕ
0.15	24.22	30.96	-146
0.2	30.96	38.6	-160
0.25	36.86	45	-172
0.33	44.7	52	-188
0.6	60.14	67	-218
1	71.53	75	-238



$$G(s) = \frac{5(1+2s)}{(1+4s)(1+0.25s)}$$



for the function $G(s) = \frac{5(1+2s)}{(1+4s)(1+0.25s)}$ - Draw the Bode plot

Sol:

The sinusoidal T.F $G(j\omega)$ is

$$G(j\omega) = \frac{5(1+j2\omega)}{(1+j4\omega)(1+0.25j\omega)}$$

Magnitude plot:

$$\omega_{c1} = \frac{1}{4} = 0.25 \text{ rad/sec} \quad \omega_{c2} = \frac{1}{2} = 0.5 \text{ rad/sec}$$

$$\omega_{c3} = \frac{1}{0.25} = 4 \text{ rad/sec}$$

$$\text{Let } \omega_l = 0.1 \text{ rad/sec}, \quad \omega_h = 10 \text{ rad/sec}$$

Term	corner freq.	slope	change in slope
5	-	0	-
$\frac{1}{1+j4\omega}$	$\omega_{c1} = \frac{1}{4} = 0.25$	-20	$0-20 = -20$
$\frac{1}{1+j2\omega}$	$\omega_{c2} = \frac{1}{2} = 0.5$	+20	$-20+20 = 0$
$\frac{1}{1+0.25j\omega}$	$\omega_{c3} = \frac{1}{0.25} = 4$	-20	$0-20 = -20$

At $\omega = \omega_l$ $M = 20 \log 5 = +14 \text{ dB}$
 At $\omega = \omega_{c1}$ $M = 20 \log 5 = +14 \text{ dB}$

At $\omega = \omega_{c2}$ $M = -20 \times \log \frac{0.5}{0.25} + 14 = +8 \text{ dB}$

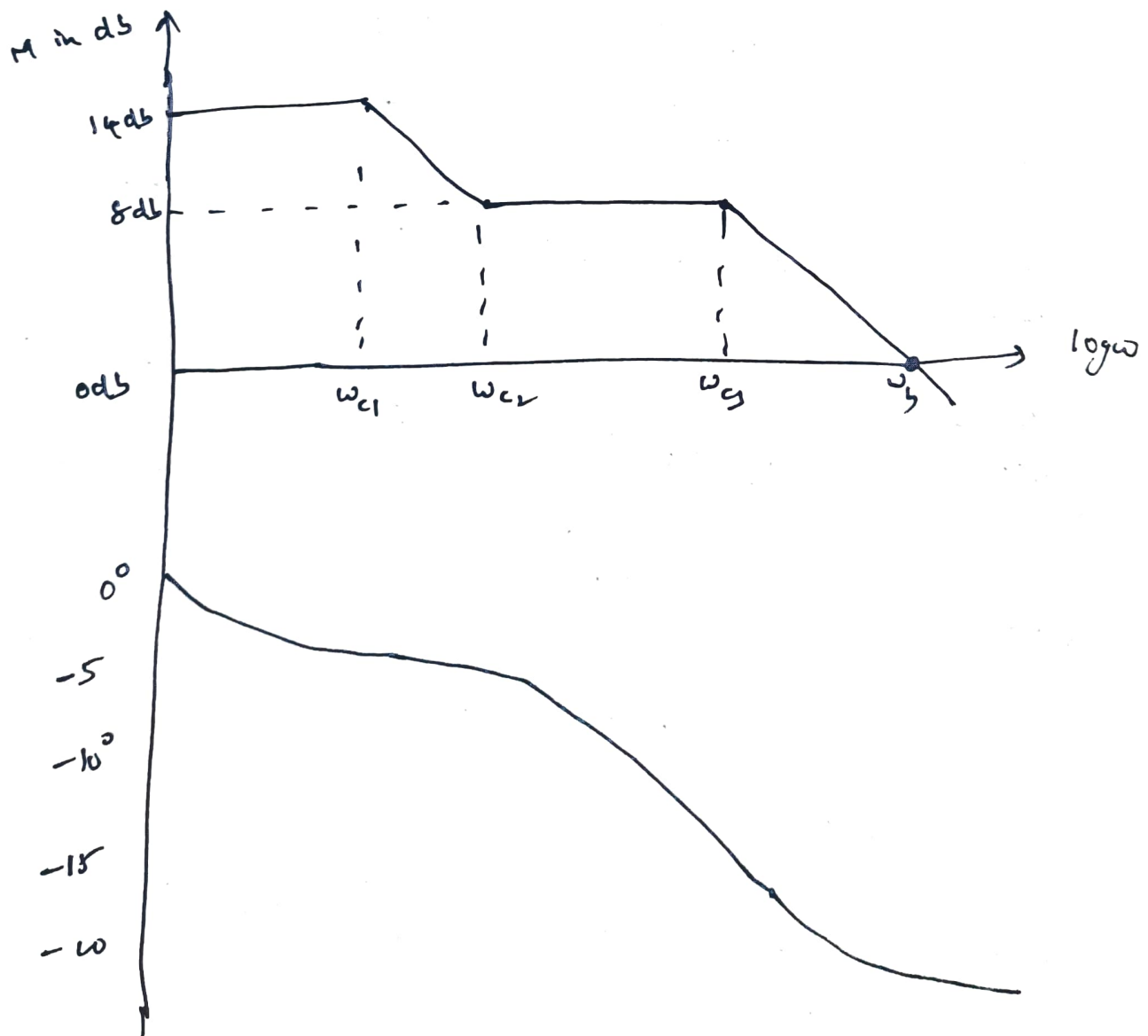
$\omega = \omega_{c3}$ $M = 0 \times \log \frac{4}{0.5} + 8 = 8 \text{ dB}$

$\omega = \omega_h$ $M = -20 \log \frac{10}{4} + 8 = 0 \text{ dB}$

phase plot:

$$\phi = +\tan^{-1}(2\omega) - \tan^{-1}(4\omega) - \tan^{-1}(0.25\omega)$$

ω	$\theta_1 = \tan^{-1} 2\omega$	$\theta_2 = \tan^{-1} 4\omega$	$\theta_3 = \tan^{-1} 0.25\omega$	$\phi = \theta_1 - \theta_2 - \theta_3$
0.1				-12
0.25				-22
0.5				-26
2				-33
4				-49
10				-70
50				-86



Given $G(s) = \frac{k \cdot e^{-0.2s}}{s(s+2)(s+8)}$ Find 'k' so that

a) gain margin = 20 dB

b) phase margin = 45°

Sol:

Let $k=1$

$$G(s) = \frac{e^{-0.2s}}{s(s+2)(s+8)}$$

$$= \frac{e^{-0.2s}}{2 \times 8 s(1+s/2)(1+s/8)}$$

$$= \frac{0.0625 \cdot e^{-0.2s}}{s(1+s/2)(1+s/8)}$$

$$G(j\omega) = \frac{0.0625 \cdot e^{-0.2j\omega}}{j\omega(1+0.5j\omega)(1+j0.125\omega)}$$

$$\begin{aligned} |0.0625 e^{-0.2j\omega}| &= 0.0625 \\ \angle 0.0625 e^{-0.2j\omega} &= -0.2\omega \end{aligned}$$

$$\begin{aligned} e^{j\theta} &\Rightarrow M = |e^{j\theta}| \\ &= |\cos\theta + j\sin\theta| \\ &= \sqrt{\cos^2\theta + \sin^2\theta} = 1 \end{aligned}$$

$$\omega_{c1} = \frac{1}{0.5} = 2 \text{ rad/sec}$$

$$\omega_{c2} = \frac{1}{0.125} = 8 \text{ rad/sec}$$

Term	Coring freq.	slope	change in slope
$\frac{0.0625}{j\omega}$	—	-20	—
$\frac{1}{1+0.5j\omega}$	$\omega_{c1} = 2$	-20	$-20 - 20 = -40$
$\frac{1}{1+0.125j\omega}$	$\omega_{c2} = 8$	-20	$-40 - 20 = -60$

Let $\omega_1 = 0.5 \text{ rad/sec}$

$\omega_h = 50 \text{ rad/sec}$

At $\omega = \omega_1$, $M = 20 \log \left| \frac{0.0625}{j\omega} \right| = 20 \log \frac{0.0625}{0.5} = -18 \text{ dB}$

$\omega = \omega_2$, $M = 20 \log \left| \frac{0.0625}{\omega} \right| = 20 \log \frac{0.0625}{2} = -30 \text{ dB}$

$\omega = \omega_{c2} \Rightarrow M = -40 \times \log 8/2 - 30 \text{ dB} = -54 \text{ dB}$

$\omega = \omega_h \Rightarrow M = -60 \times \log 50/8 - 54 = -102 \text{ dB}$

phase plots

$\phi = -0.2 \omega \times \frac{180^\circ}{\pi} - 90^\circ - \tan^{-1} 0.5 \omega - \tan^{-1} 0.125 \omega$

Unit - IV

Stability Analysis in Frequency domain

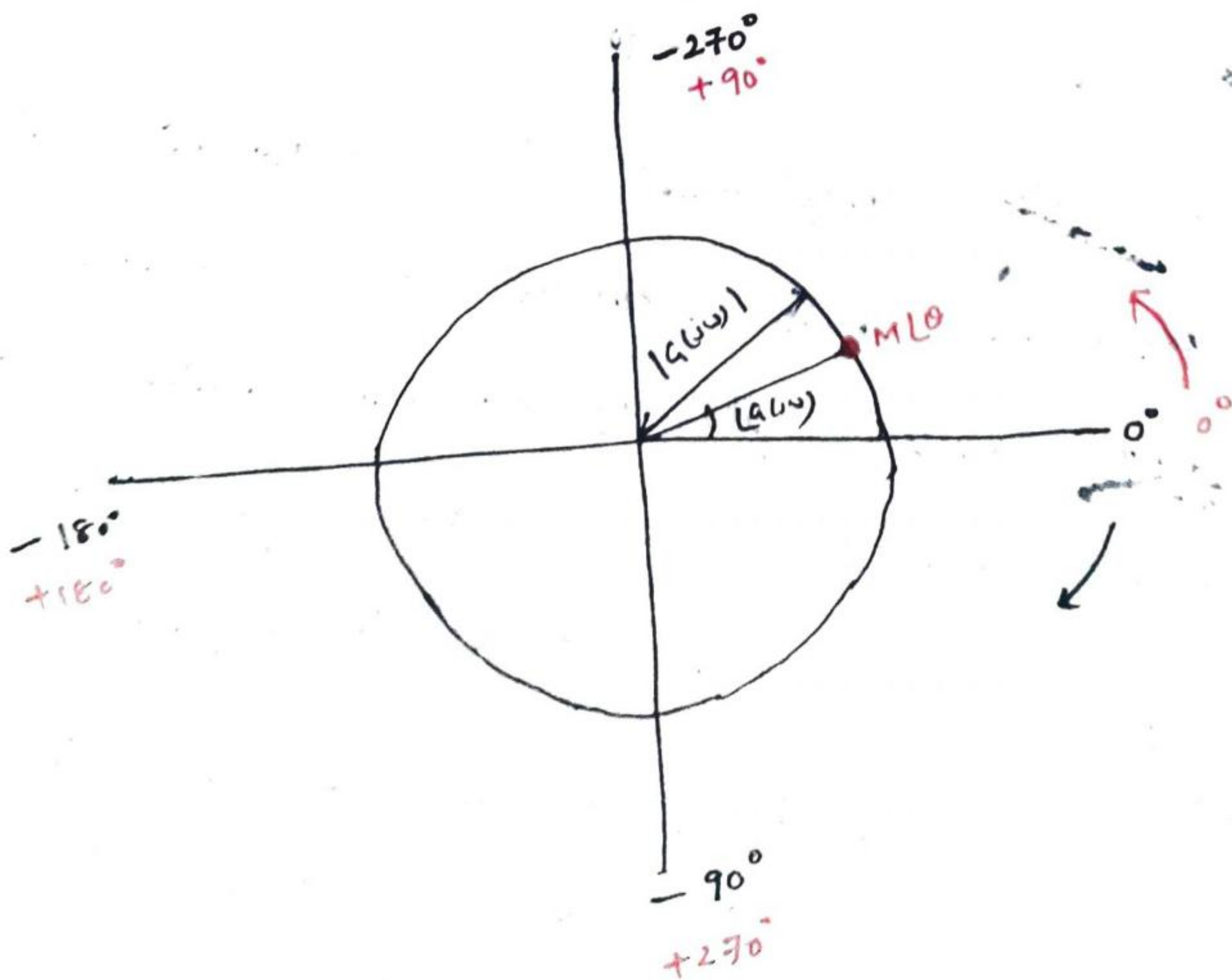
Polar plot:

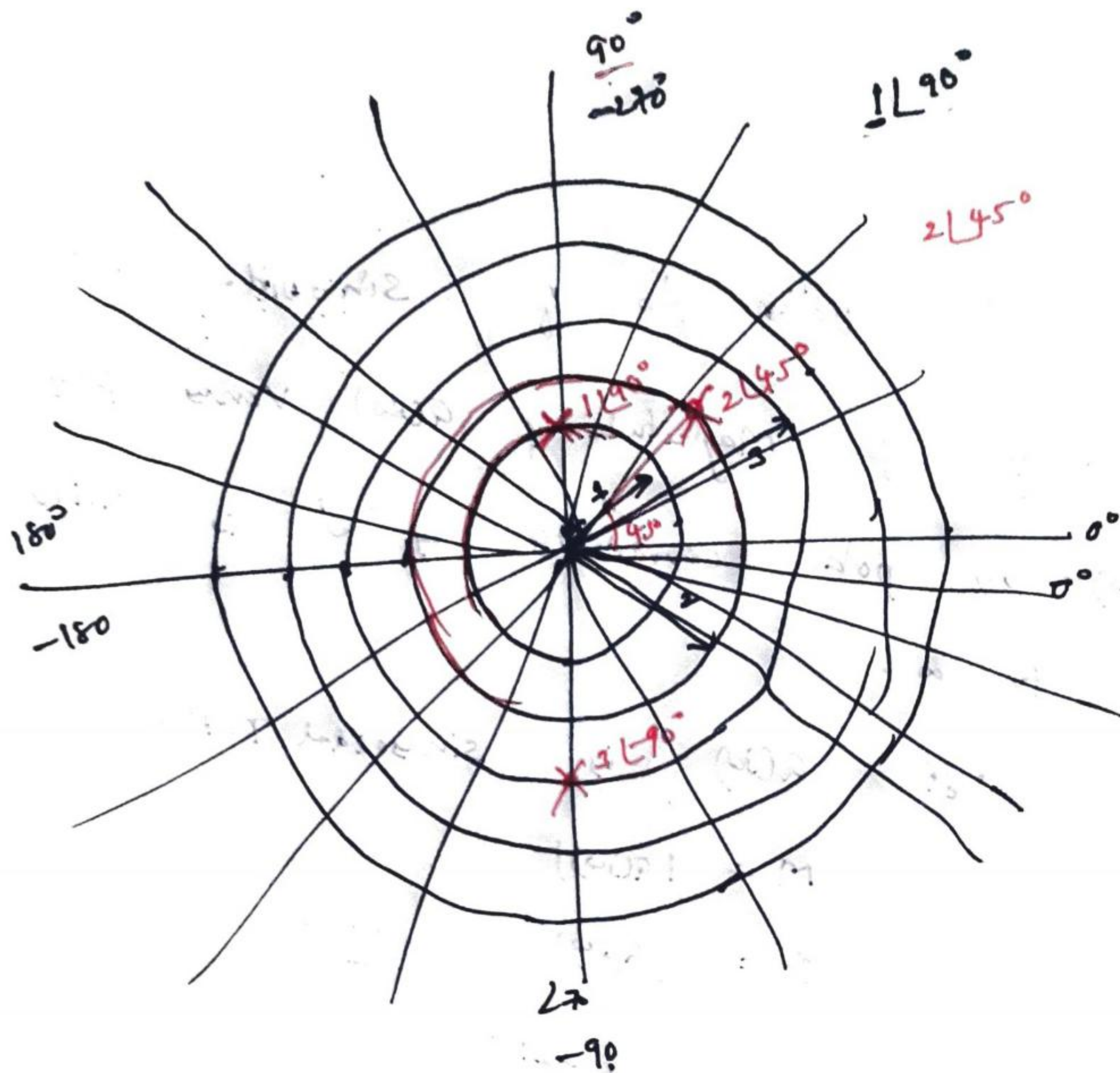
The polar plot of sinusoidal T.F $G(j\omega)$ is a plot of Magnitude of $G(j\omega)$ versus phase angle of $G(j\omega)$ on polar coordinates as ω is varied from 0 to ∞ .

Let $G(j\omega)$ is sinusoidal T.F.

$$M = |G(j\omega)|$$

$$\theta = \angle G(j\omega) \quad \underline{M\angle\theta}$$





The polar graph sheet has concentric circles and radial lines. The circles represent the magnitude and the radial lines represent phase angles. The magnitude of a point is given by the value of circle passing through that point and phase angle given by radial line passing through that point.

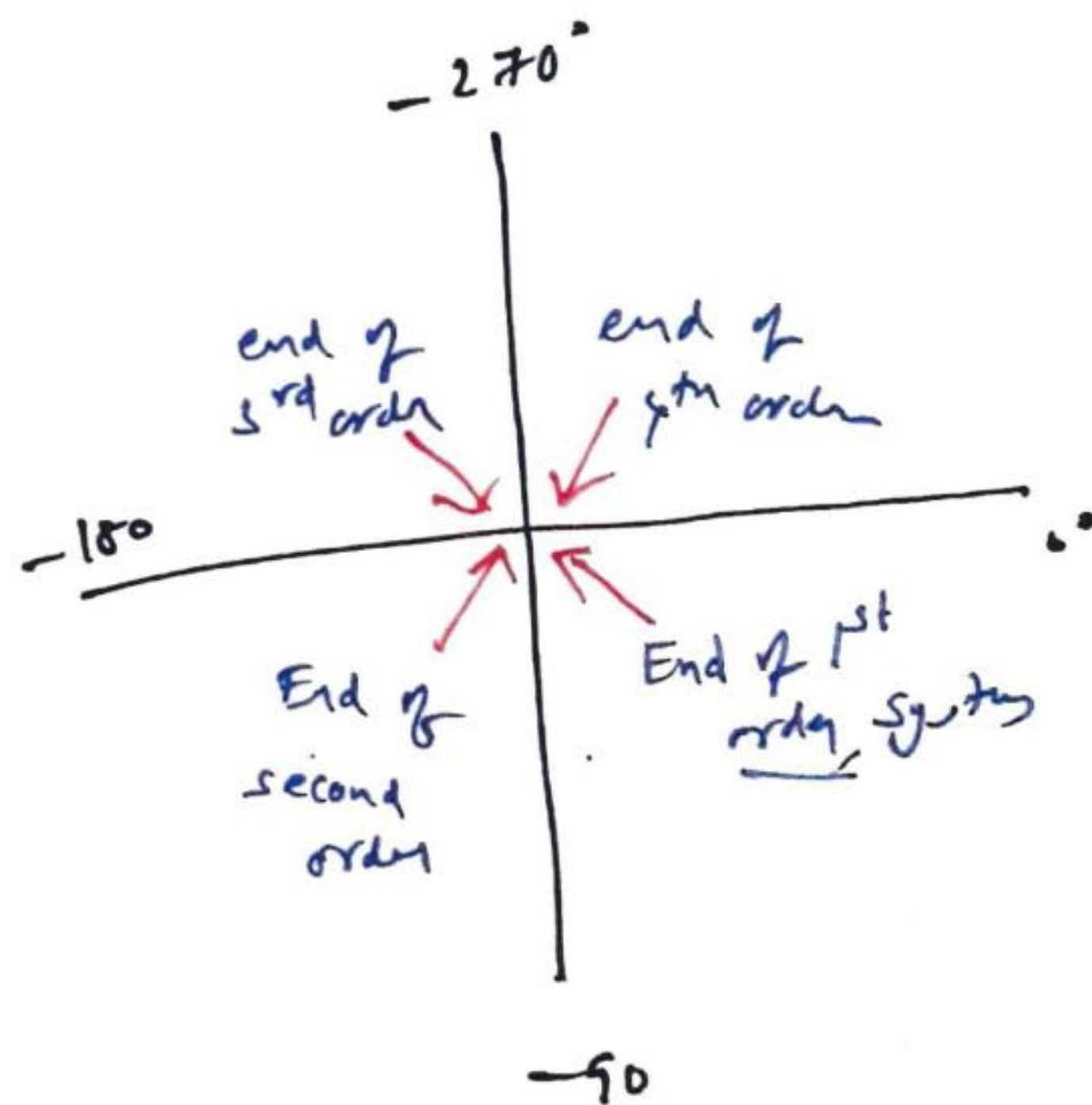
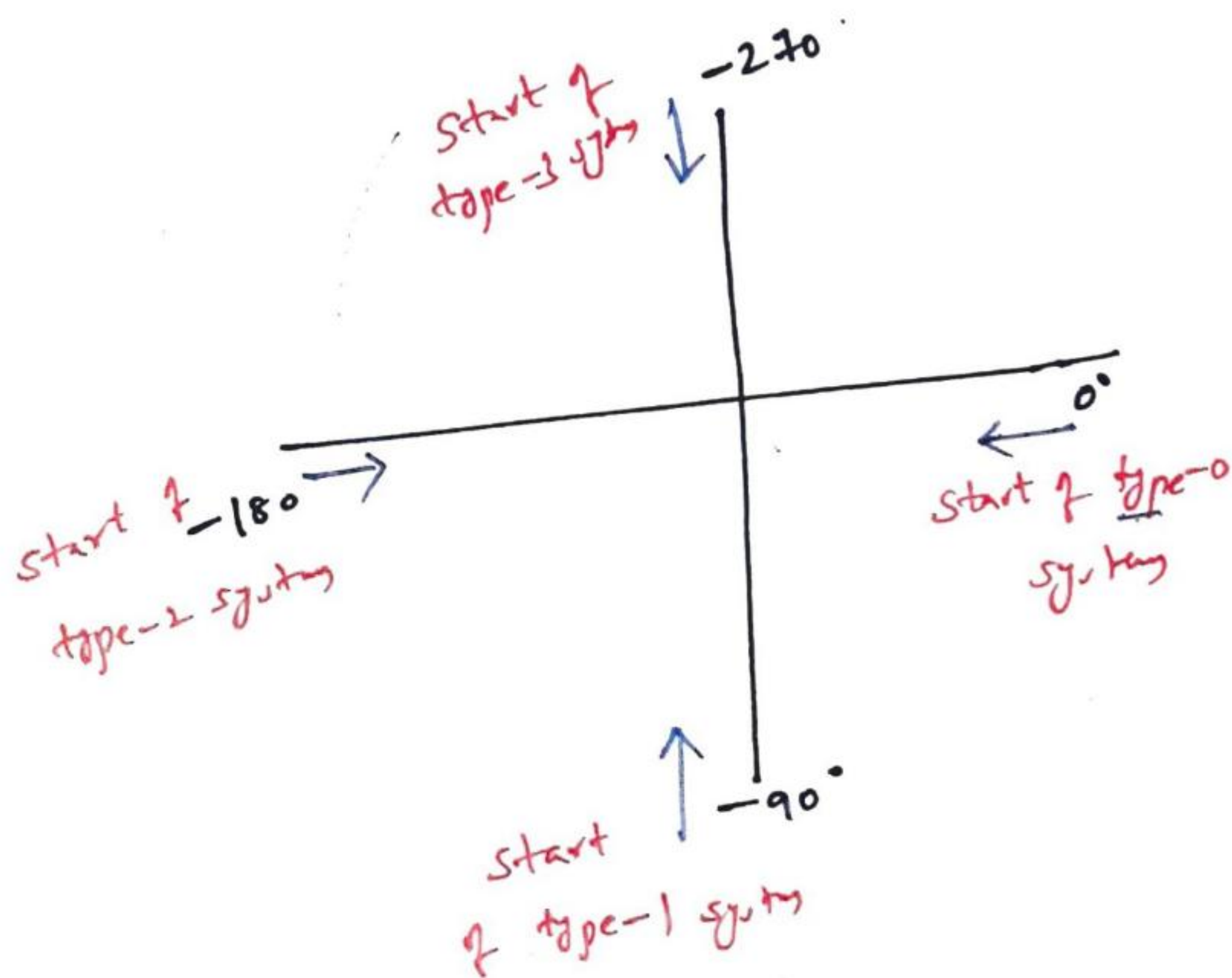
positive phase angle $\longrightarrow$ anti clockwise
 negative " $\longrightarrow$ clockwise

procedure:

In order to plot polar plot, the magnitude and phase angle $\angle G(j\omega)$ are computed for various values of ω and tabulated. Fix all the points on polar graph sheet and join the points.

Minimum phase S.F.:

It is the system with all poles and zeros on left half of s-plane. The start and end points of minimum phase systems are shown in fig.



In minimum phase system, type number of system determines the quadrant at which polar plot starts and the order determines quadrant at which polar plot ends.

1)

$$G(s) = \frac{1}{1+sT}$$

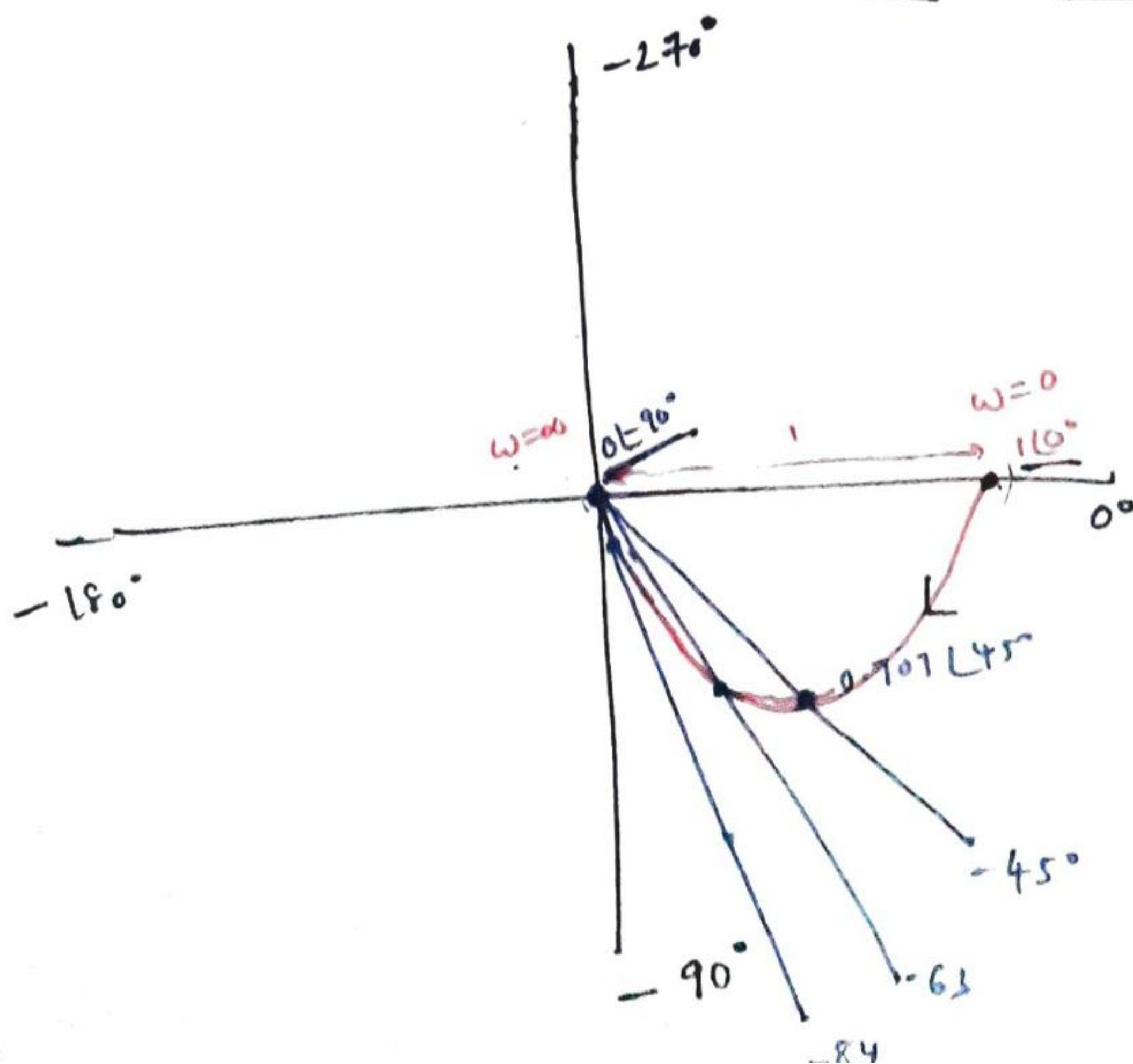
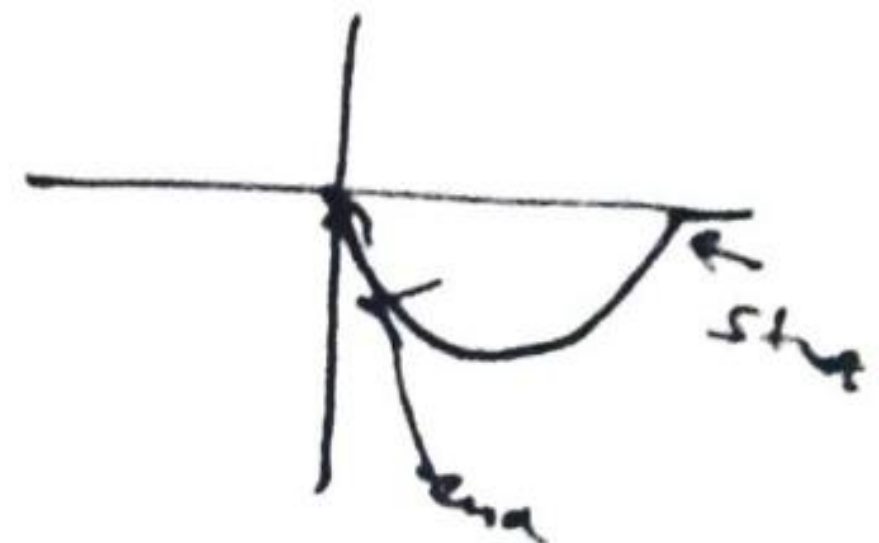
→ order - 1
Type - 0

$$G(j\omega) = \frac{1}{1+j\omega T}$$

$$M = |G(j\omega)| = \frac{1}{\sqrt{1+\omega^2 T^2}}$$

$$\theta = \angle G(j\omega) = -\tan^{-1}(\omega T)$$

ω	M	θ
0	1	0°
$\frac{1}{T}$	0.707	-45°
$\frac{2}{T}$	0.44	-63.43°
$\frac{10}{T}$	0.1	-84°
$\vdots$	$\vdots$	$\vdots$
∞	0	-90°



Procedure to draw the polar plot:

- 1) Convert given T.F into sinusoidal T.F.
- 2) Find Magnitude and phase angle at $\omega=0$ and $\omega=\infty$. Let the magnitudes M_1, M_2 phase angles ϕ_1, ϕ_2 respectively

$$\text{At } \omega=0, \quad M_1 \angle \phi_1$$

$$\text{At } \omega=\infty, \quad M_2 \angle \phi_2$$

To find ending direction:

If $\phi_1 - \phi_2 =$ positive then clockwise direction
= negative, anticlockwise direction

→ starting direction is 'consider whenever T.F consists of finite poles and finite zeros, which are having positive sign terms.

If the finite pole is near to origin then starting direction is clockwise. If the finite zero is near to origin then starting direction is anticlockwise

Eg:

$$G(s) = \frac{1}{s(1+sT)}$$

$$G(j\omega) = \frac{1}{j\omega(1+j\omega T)}$$

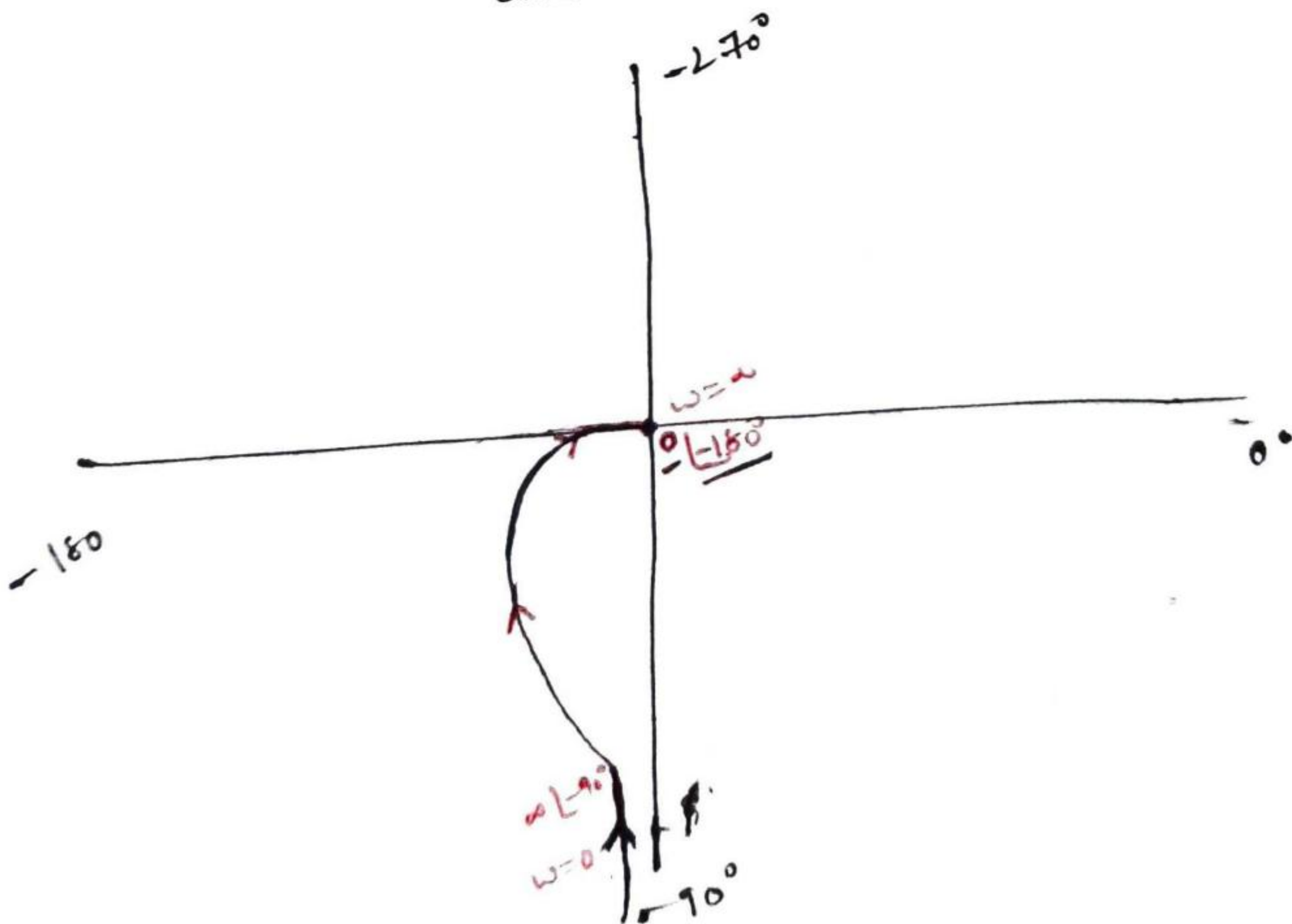
$$M = \frac{1}{\omega \sqrt{1+\omega^2 T^2}}$$

$$\phi = -90^\circ - \tan^{-1}(\omega T)$$

$$\text{As } \omega \rightarrow 0, \quad M_1 = \infty, \quad \phi_1 = -90^\circ$$

$$\text{As } \omega \rightarrow \infty, \quad M_2 = 0, \quad \phi_2 = -90^\circ - 90^\circ = -180^\circ$$

Ending direction $\phi_1 - \phi_2 = -90^\circ + 180^\circ = 90^\circ$ positive.
↓
clockwise.



$$G(s) = \frac{1}{(1+sT_1)(1+sT_2)}$$

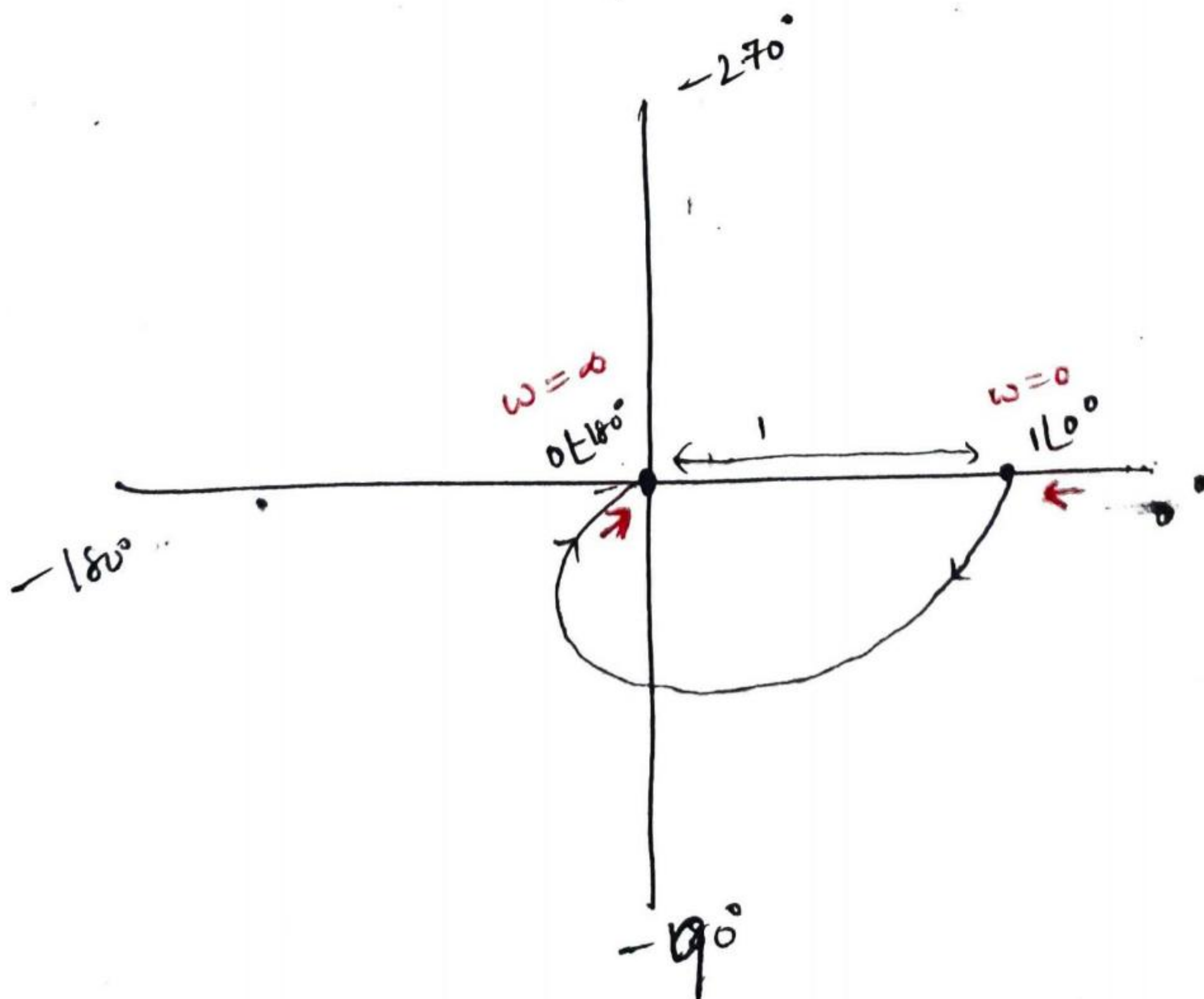
$$G(j\omega) = \frac{1}{(1+j\omega T_1)(1+j\omega T_2)}$$

$$M = \frac{1}{\sqrt{1+\omega^2 T_1^2} \sqrt{1+\omega^2 T_2^2}}$$

$$\phi = -\tan^{-1}(\omega T_1) - \tan^{-1}(\omega T_2)$$

At $\omega=0$, $M_1 = 1$, $\phi_1 = 0^\circ$
 $\omega=\infty$, $M_2 = 0$, $\phi_2 = -180^\circ$

E.D $\Rightarrow \phi_1 - \phi_2 = 0 - (-180^\circ) = +180^\circ$
 clock wise



Type - 0
 order - 2

$$G(s) = \frac{1}{(1+sT_1)(1+sT_2)(1+sT_3)}$$

$$G(j\omega) = \frac{1}{(1+j\omega T_1)(1+j\omega T_2)(1+j\omega T_3)}$$

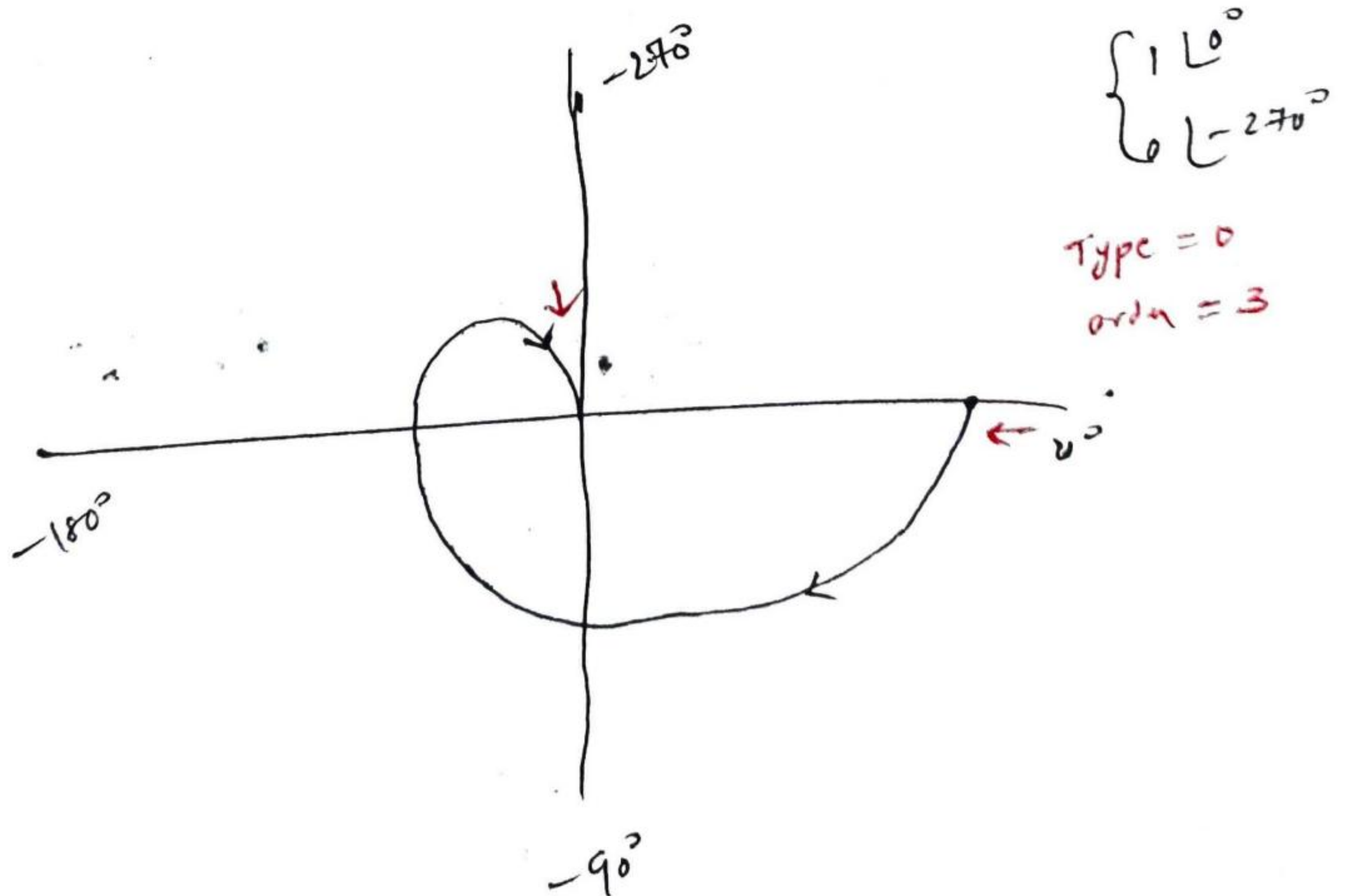
$$M = \frac{1}{\sqrt{1+\omega^2 T_1^2} \sqrt{1+\omega^2 T_2^2} \sqrt{1+\omega^2 T_3^2}}$$

$$\phi = -\tan^{-1}(\omega T_1) - \tan^{-1}(\omega T_2) - \tan^{-1}(\omega T_3)$$

$$\omega = 0, \quad M_1 = 1, \quad \phi_1 = 0^\circ$$

$$\omega = \infty, \quad M_2 = 0, \quad \phi_2 = -270^\circ$$

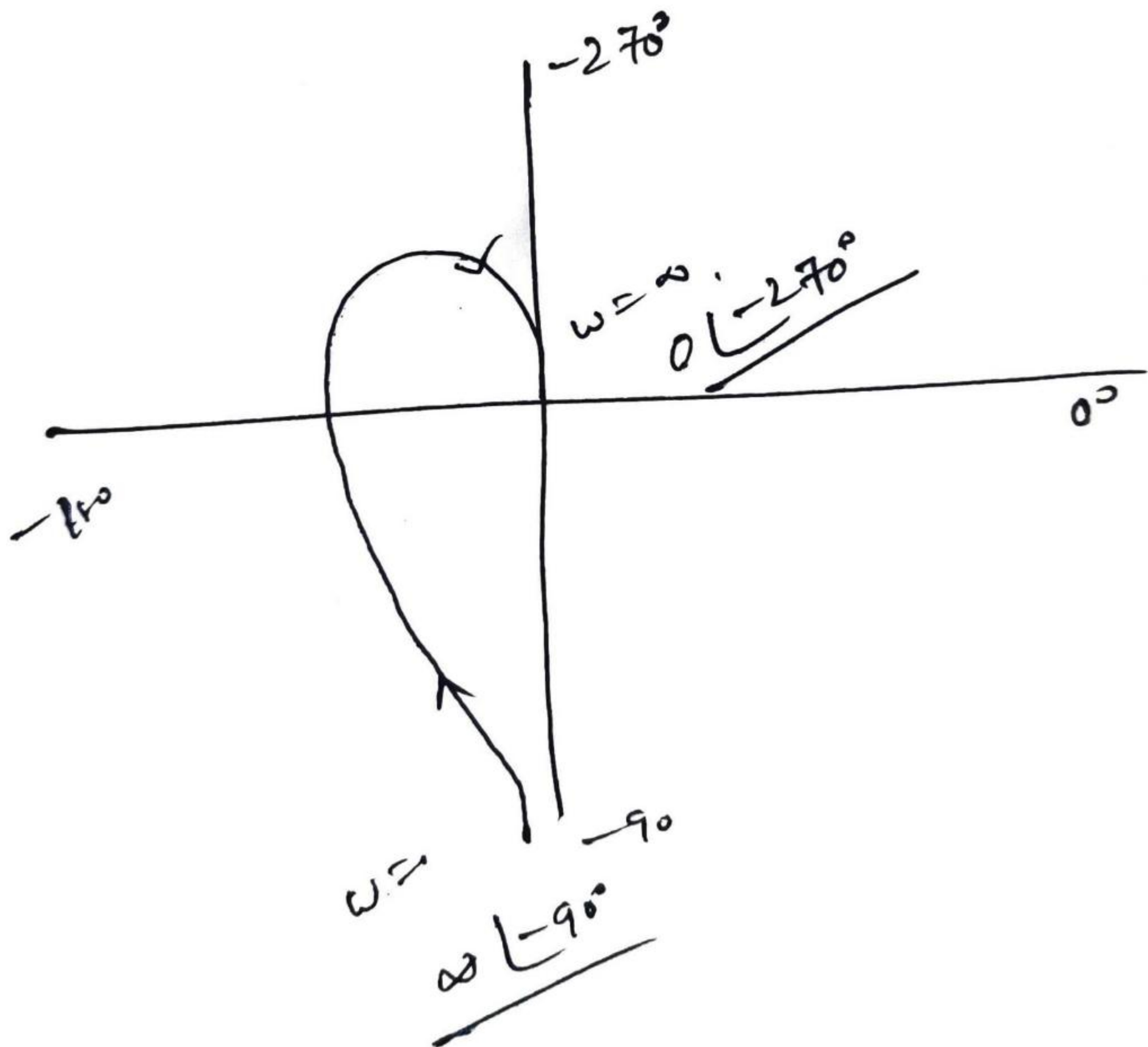
$$\text{E.D.} \Rightarrow \phi_1 - \phi_2 = 0 - (-270^\circ) = 270^\circ \text{ (clockwise)}$$



$$G(s) = \frac{1}{s(1+sT_1)(1+sT_2)}$$

$$\text{Type} = 1 \checkmark$$

$$\text{order} = 3 \checkmark$$



$$G(s) = \frac{1}{s^2(1+sT_1)(1+sT_2)}$$

$$G(j\omega) = \frac{1}{(j\omega)^2(1+j\omega T_1)(1+j\omega T_2)}$$

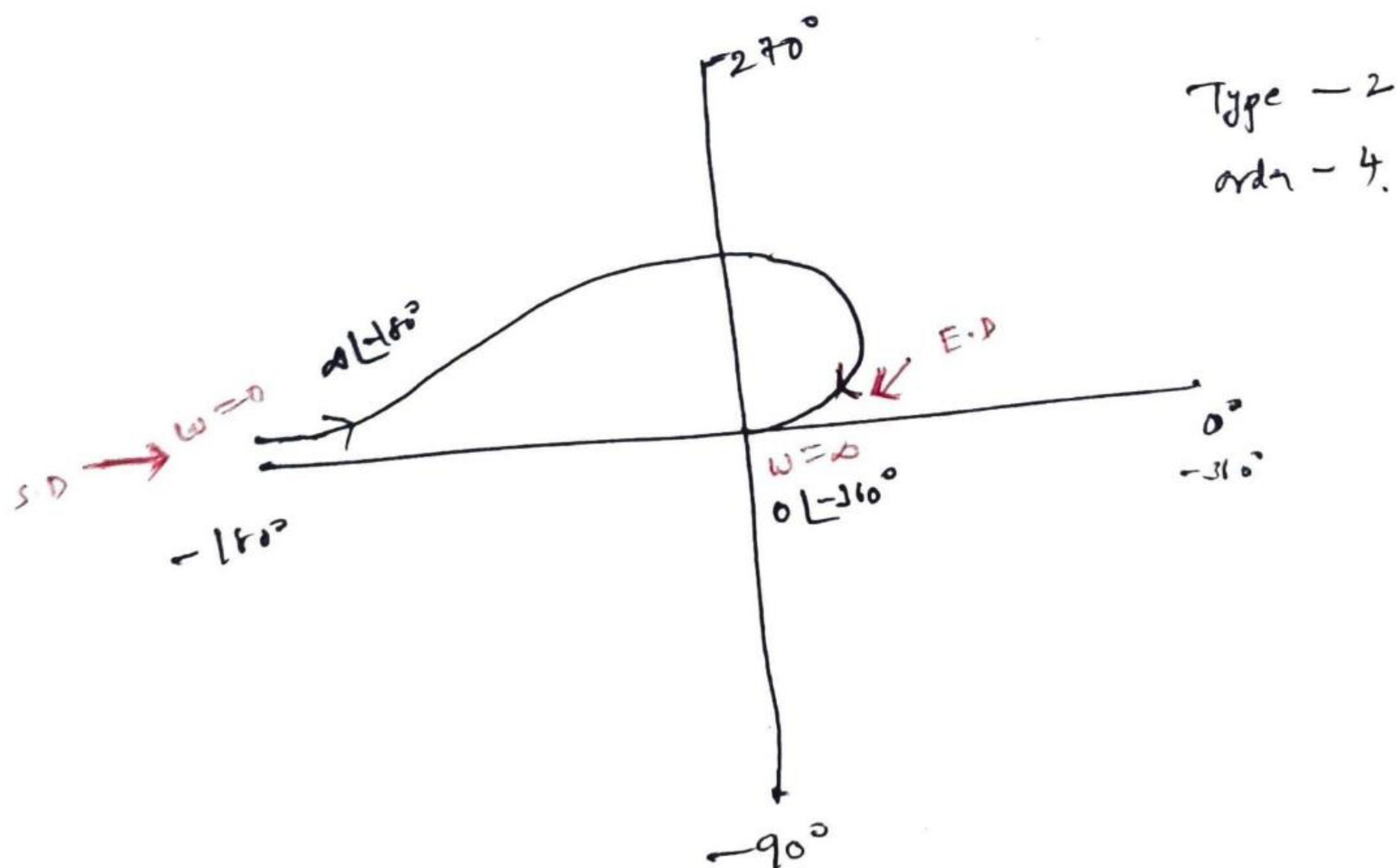
$$M = \frac{1}{\omega^2 \sqrt{1+\omega^2 T_1^2} \sqrt{1+\omega^2 T_2^2}}$$

$$\phi = -180^\circ - \tan^{-1}(\omega T_1) - \tan^{-1}(\omega T_2)$$

As $\omega = 0$, $M_1 = \infty$, $\phi_1 = -180^\circ$
 $\omega = \infty$, $M_2 = 0$, $\phi_2 = -360^\circ$

$$E.D \Rightarrow \phi_1 - \phi_2 = -180^\circ - (-360^\circ) = +180^\circ = \text{positive}$$

clockwise



$$G(s) = \frac{1}{s^2 (1+sT_1) (1+sT_2) (1+sT_3)}$$

$$M = \frac{1}{\omega^2 \sqrt{1+(\omega T_1)^2} \sqrt{1+\omega^2 T_2^2} \sqrt{1+\omega^2 T_3^2}}$$

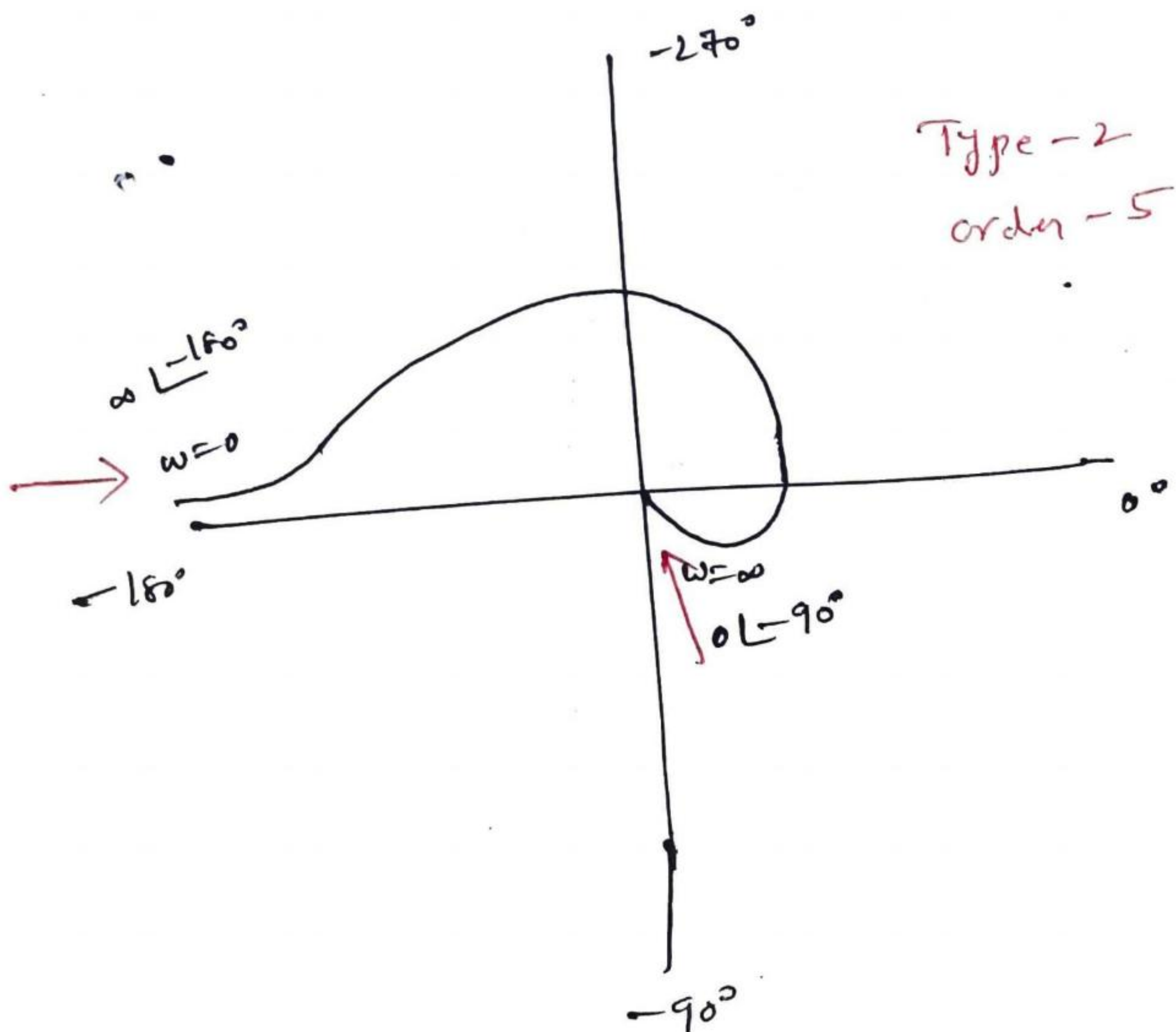
$$\phi = -180^\circ - \tan^{-1}(\omega T_1) - \tan^{-1}(\omega T_2) - \tan^{-1}(\omega T_3)$$

$$\omega = 0, \quad M_1 = \infty, \quad \phi_1 = -180^\circ$$

$$\omega = \infty, \quad M_2 = 0, \quad \phi_2 = -450^\circ \text{ (or)} \underline{\underline{-90^\circ}}$$

$$\text{E.D} \Rightarrow \phi_1 - \phi_2 = -180^\circ + 450^\circ = 270^\circ$$

clockwise.



$$G(s) = \frac{1}{s}$$

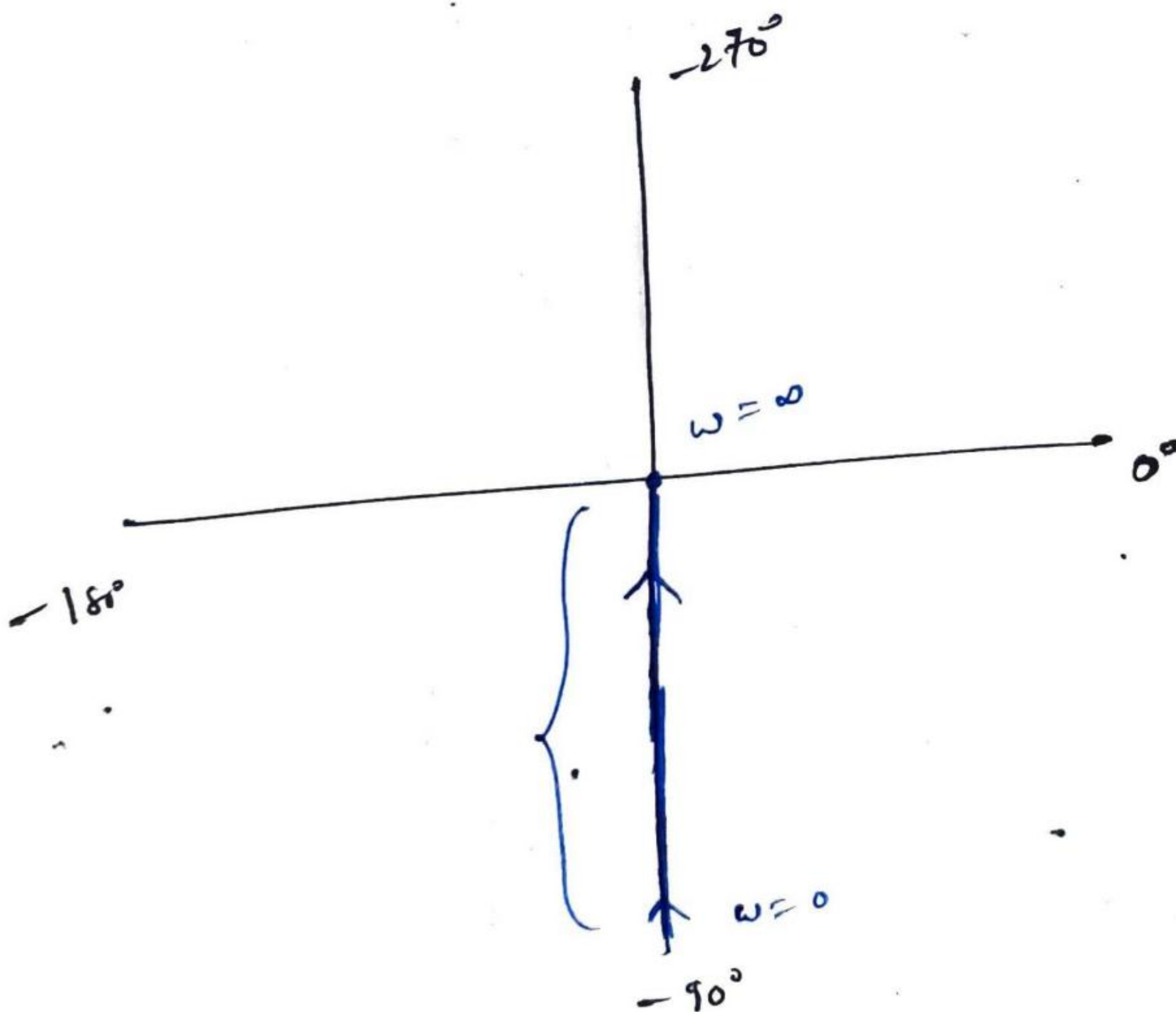
$$G(j\omega) = \frac{1}{j\omega}$$

$$M = \frac{1}{\omega}$$

$$\phi = -90^\circ$$

$$\omega = 0 \Rightarrow M_1 = \infty, \phi_1 = -90^\circ$$

$$\omega = \infty \Rightarrow M_2 = 0, \phi_2 = -90^\circ$$



$$G(s) = s$$

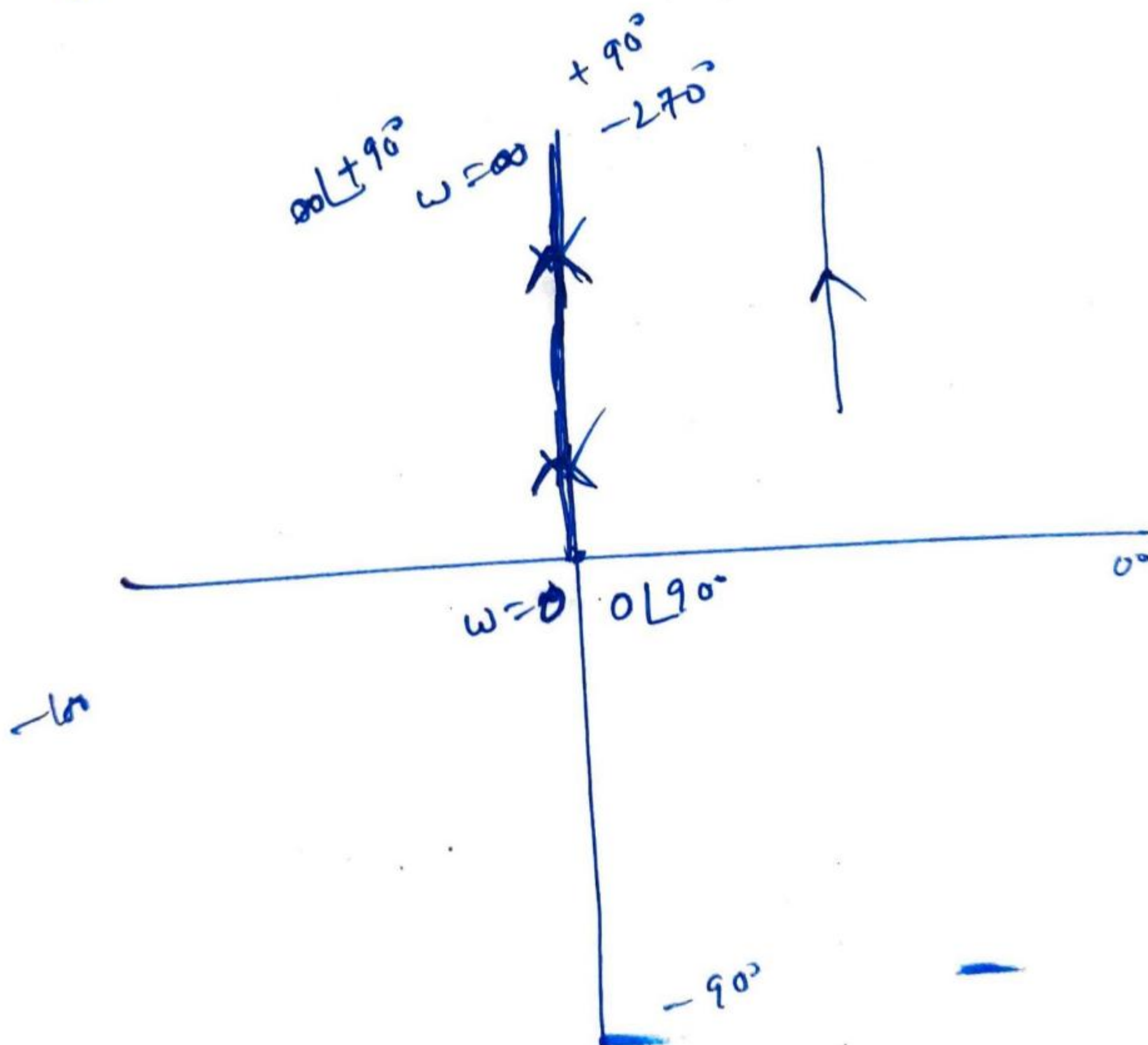
$$G(j\omega) = j\omega$$

$$M = \omega$$

$$\phi = +90^\circ$$

$$\omega = 0 \Rightarrow M_1 = 0, \phi = 90^\circ$$

$$\omega = \infty \Rightarrow M_2 = \infty, \phi = 90^\circ$$

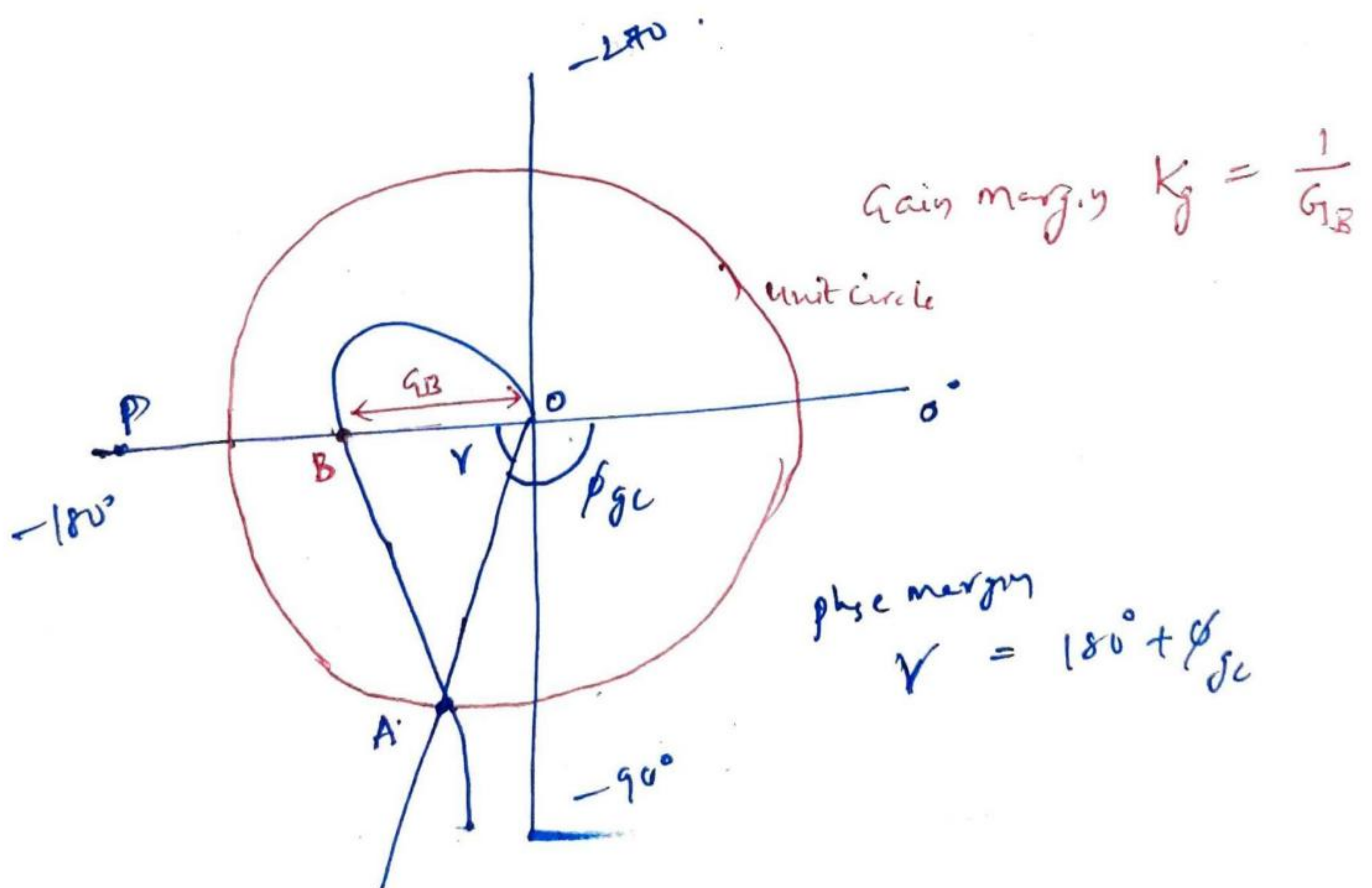


Determination of gain margin and phase margin from polar plot:

The gain margin is defined as the inverse of magnitude of $G(j\omega)$ at phase crossover frequency (ω_{pc}). The ω_{pc} is the frequency at which phase $+180^\circ$.

Let the polar plot cut the 180° axis at point 'B' and the magnitude circle passing through the point 'B' is G_B . Then

$$\text{Gain margin } K_g = \frac{1}{G_B}$$

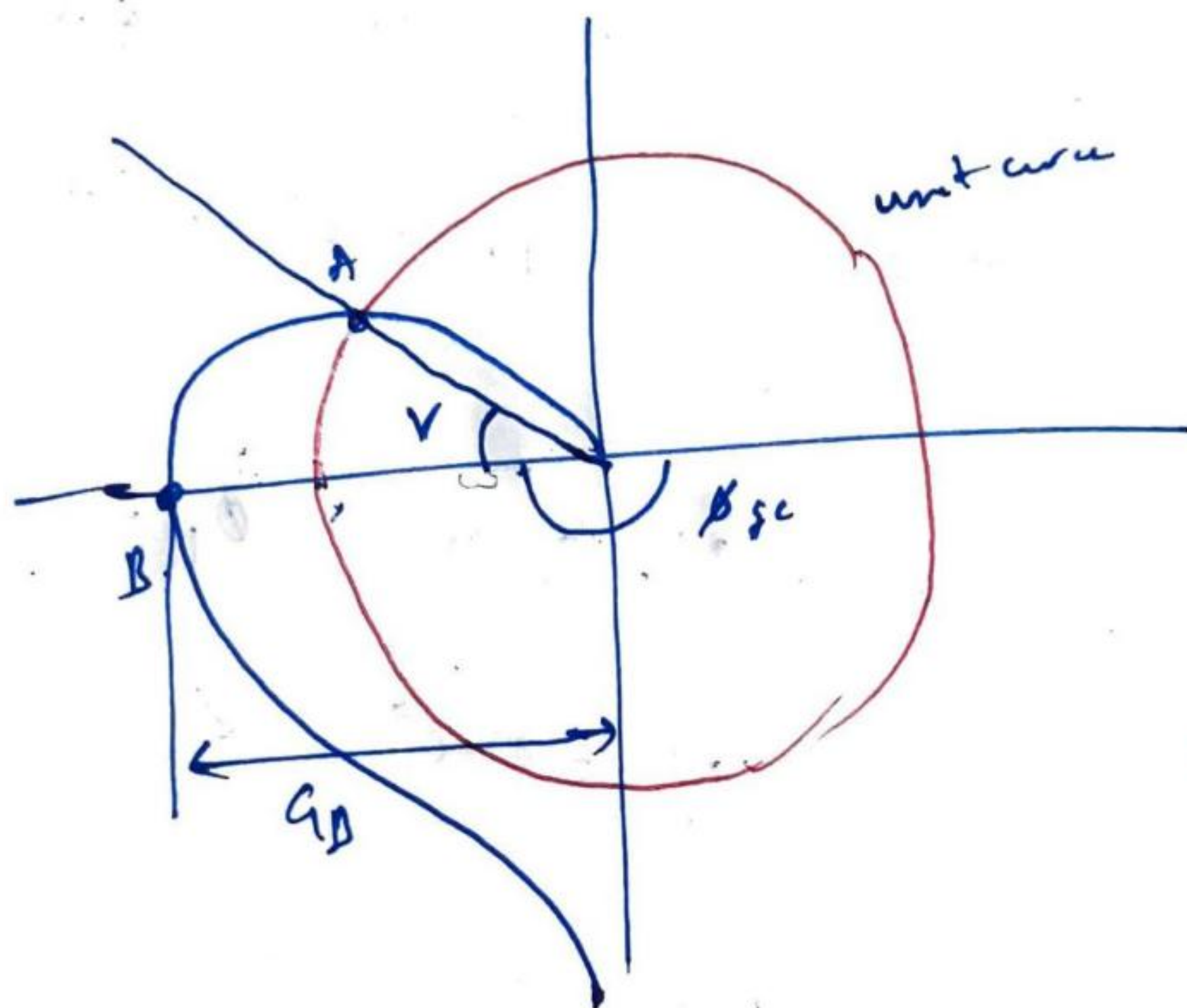


If $\angle OP < -180^\circ$ PM is positive
 $> -180^\circ$ PM is negative

The phase margin is defined as

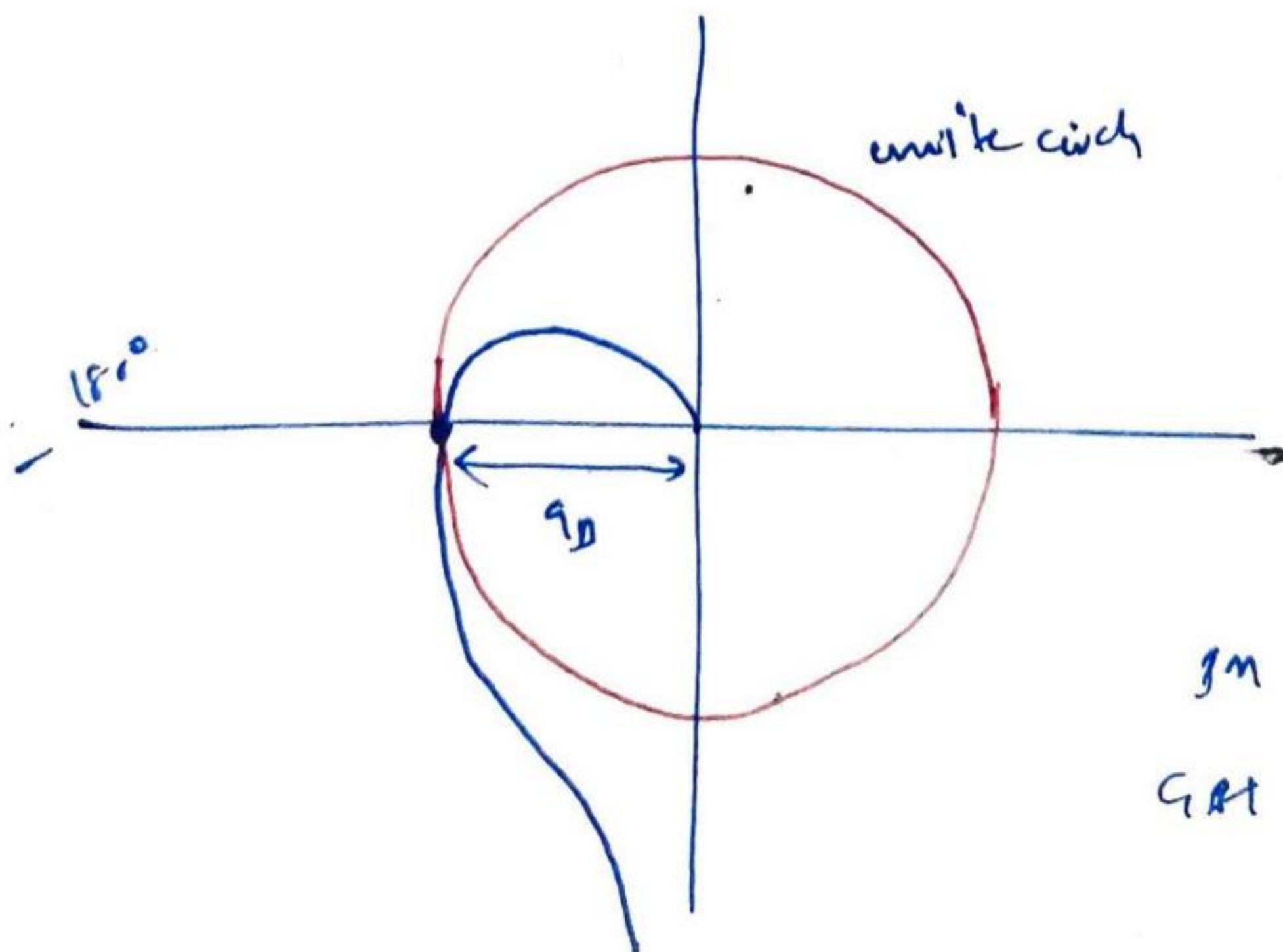
$$\gamma = 180^\circ + \phi_{gc}$$

where ϕ_{gc} is phase of $G(s)$ at ω_{gc} . The ω_{gc} is the freq at which gain is 0 dB (or) unity.



$$\omega_{gc} > \omega_{gc}$$

PM $\rightarrow$ positive
GM $\rightarrow$ stable



$$\omega_{gc} = \omega_{gc}$$

Marginally stable

$$\gamma = 180^\circ$$

$$G_M = 0 \text{ dB}$$

$$G(s) = \frac{1}{s(1+s)(1+2s)}$$

Sketch the polar plot and determine GM & PM

$$G(j\omega) = \frac{1}{j\omega(1+j\omega)(1+2j\omega)}$$

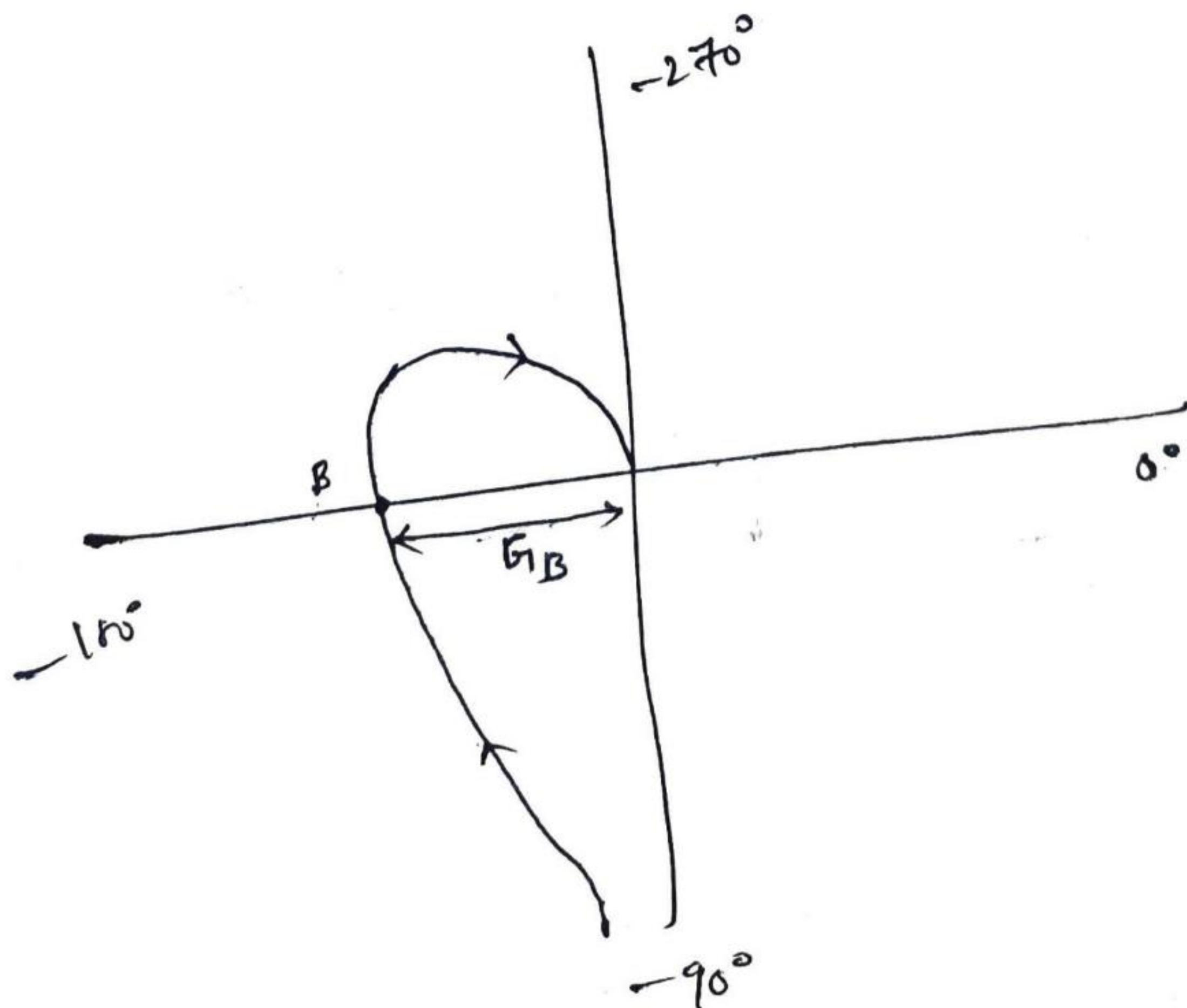
$$M = \frac{1}{\omega \sqrt{1+\omega^2} \sqrt{1+4\omega^2}}$$

$$\phi = -90^\circ - \tan^{-1}\omega - \tan^{-1}2\omega$$

$$\omega=0 \Rightarrow M_1 = \infty \quad \phi_1 = -90^\circ$$

$$\omega=\infty \Rightarrow M_2 = 0^\circ \quad \phi_2 = -270^\circ$$

E.D. $\Rightarrow$ clockwise



Intersecting point with negative ^{Real} ~~imag~~ axis

$$\angle G(s) + (s) = -180^\circ$$

$$-90^\circ - \tan^{-1} \omega - \tan^{-1} 2\omega = -180^\circ$$

$$-\tan^{-1} \omega - \tan^{-1} 2\omega = -90^\circ$$

$$\tan^{-1} \left[\frac{\omega + 2\omega}{1 - 2\omega^2} \right] = +90^\circ$$

$$\frac{\omega + 2\omega}{1 - 2\omega^2} = \frac{1}{0}$$

$$1 - 2\omega^2 = 0$$

$$2\omega^2 = 1$$

$$\boxed{\omega_{pc} = \frac{1}{\sqrt{2}}}$$

Magnitude at $\omega_{pc} = \frac{1}{\sqrt{2}}$

$$M = \frac{1}{\omega \sqrt{1+\omega^2} \sqrt{1+4\omega^2}}$$

$$\begin{aligned} M \Big|_{\omega=\omega_{pc}} &= \frac{1}{\frac{1}{\sqrt{2}} \sqrt{1+\frac{1}{2}} \sqrt{1+4\frac{1}{2}}} \\ &= \frac{\sqrt{2} \sqrt{2}}{3} = \frac{2}{3} \end{aligned}$$

$$G.M = \frac{1}{|G(s)|_{\omega=\omega_{pc}}} = \frac{3}{2}$$

To find ~~Gain~~ phase margin!

step 1: calculate ω_{gc} by equating $M = 1$

step 2: calculate phase angle at ω_{gc} i.e. ϕ_{gc}

step 3: calculate phase margin

$$\gamma = 180^\circ + \phi_{gc}$$

From Given OLTF

$$M = \frac{1}{\omega \sqrt{1+\omega^2} \sqrt{1+4\omega^2}}$$

$$1 = \frac{1}{\omega \sqrt{1+\omega^2} \sqrt{1+4\omega^2}}$$

$$\omega(1+\omega^2)(1+4\omega^2) = 1$$

$$\omega_{gc} = \underline{\hspace{2cm}}$$

$$\text{phase angle } \phi \Big|_{\omega=\omega_{gc}} = -90^\circ - \tan^{-1} \omega_{gc} - \tan^{-1}(2\omega_{gc})$$

$$\phi_{gc} = \underline{\hspace{2cm}}$$

$$\gamma = 180^\circ + \phi_{gc}$$

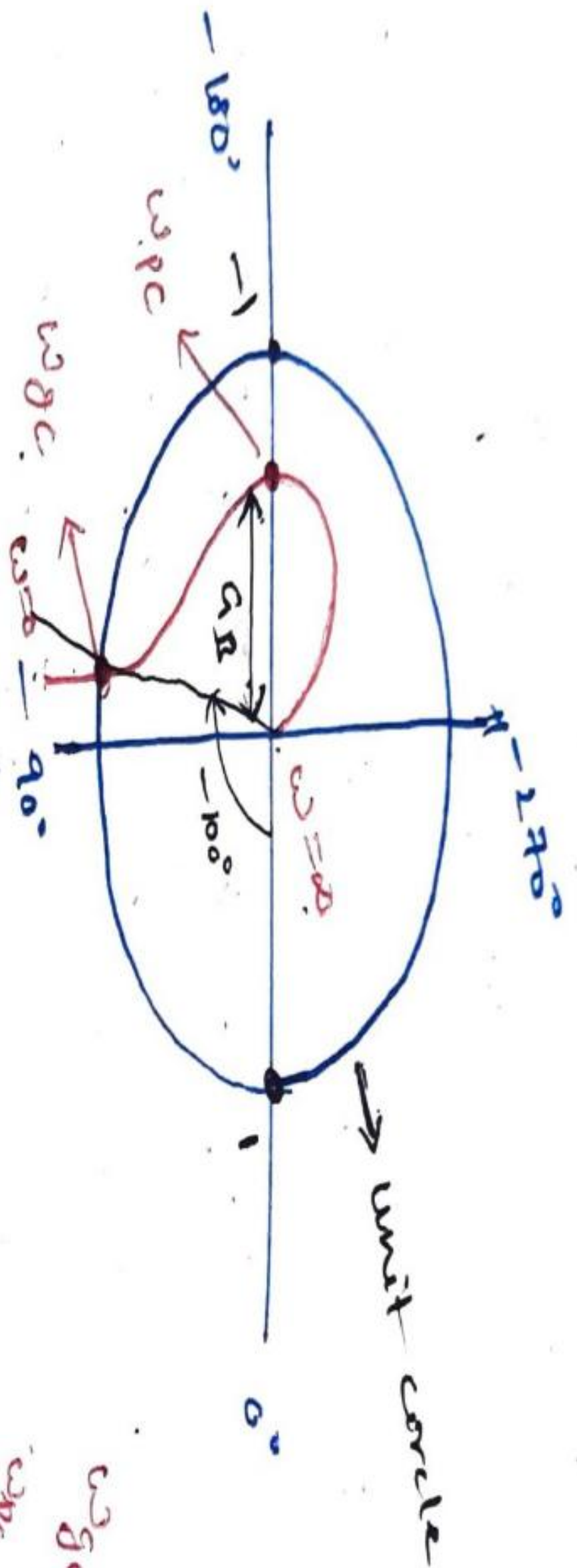
$$= \underline{\hspace{2cm}} + \underline{12^\circ}$$

Stability Analysis using polar plot:-

$$-1 + j0$$

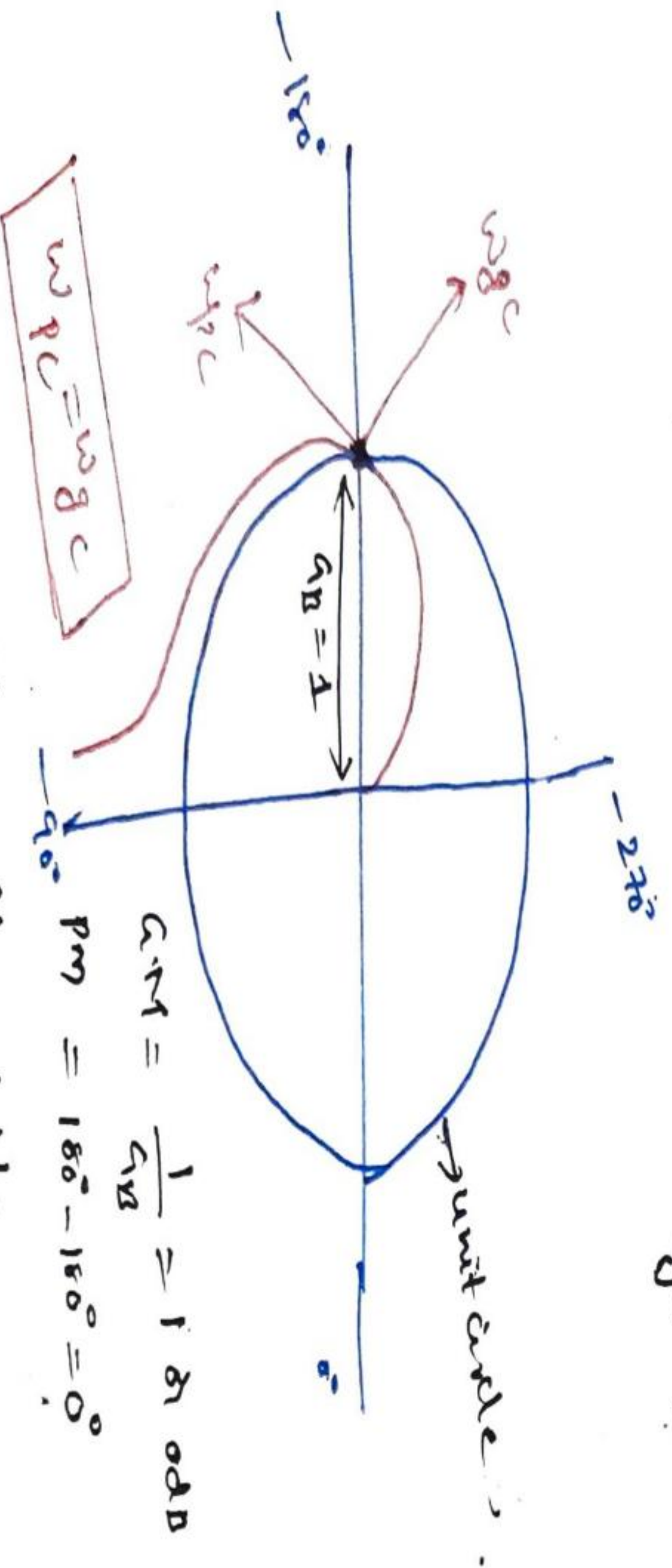
$$1 + G(s)H(s) = 0$$

$$\frac{G(s) \cdot H(s)}{1 + G(s)H(s)} = \frac{-1 + j0}{\text{Ref. point}}$$



$$\omega_{gc} < \omega_{pc}$$

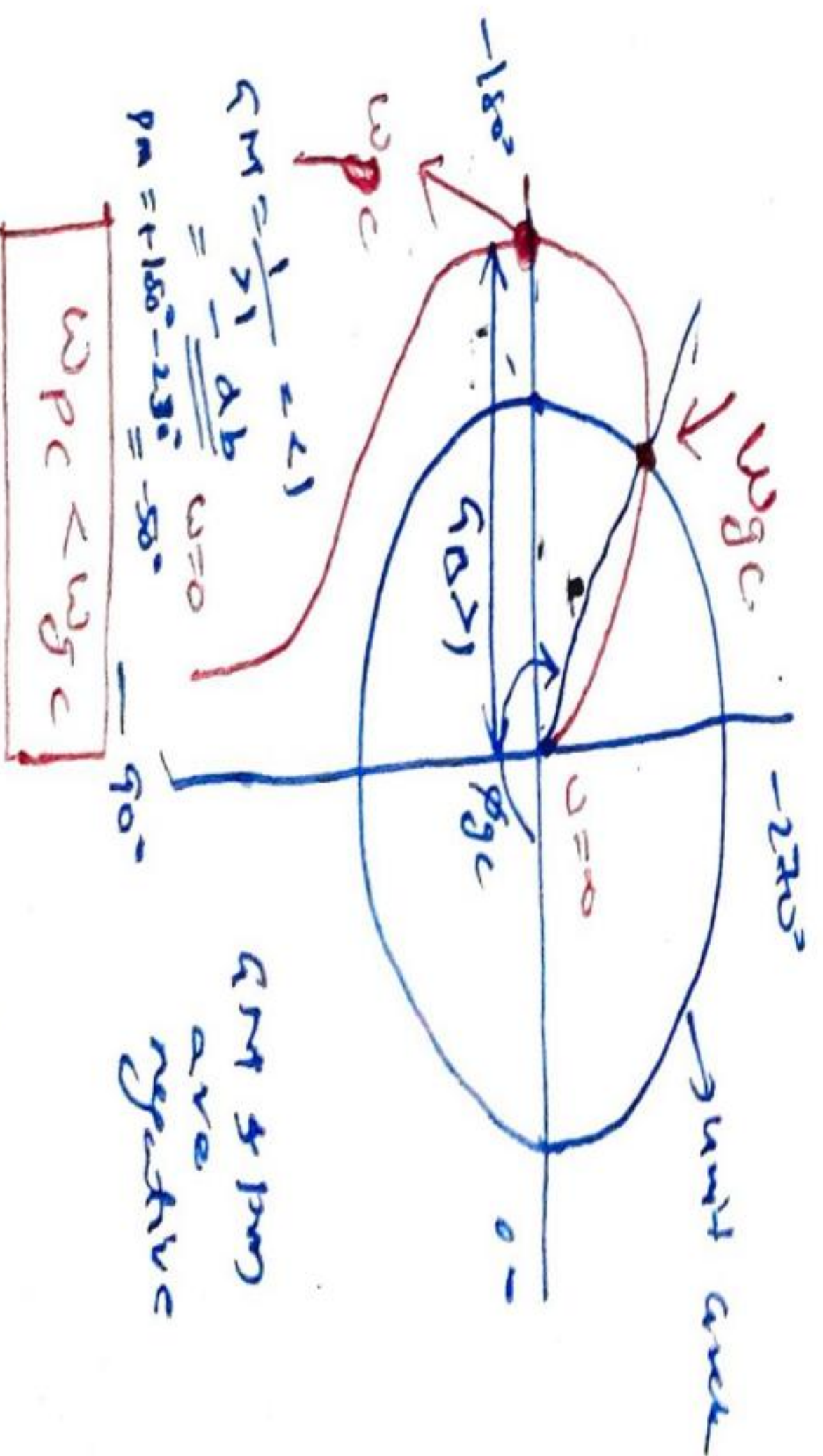
System is stable



Marginally stable

$$GM = \frac{1}{|G(j\omega)|_{\omega_{pc}}} = \frac{1}{G_{12}} > 1$$

$GM = 180^\circ + \phi_{gc} = 180^\circ - 100^\circ = 80^\circ$ Positive



System is unstable

→ sketch polar plot

$$G(s) = \frac{(s+1)}{s^2(s+2)(s+3)}$$

sol:

$$G(j\omega) = \frac{(j\omega+1)}{(j\omega)^2(j\omega+2)(j\omega+3)}$$

$$M = \frac{\sqrt{\omega^2+1}}{\omega^2 \sqrt{\omega^2+4} \sqrt{\omega^2+9}}$$

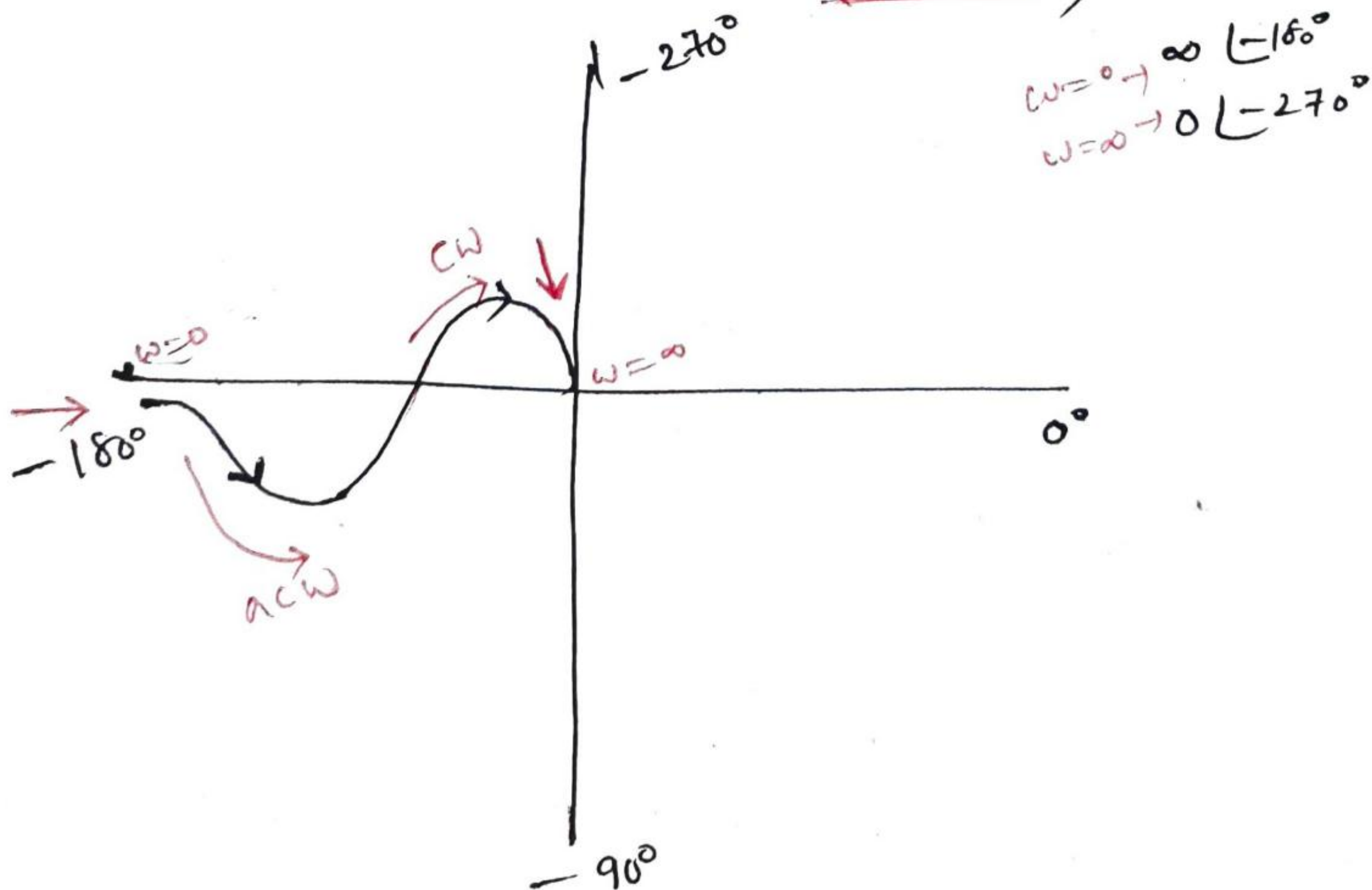
$$\phi = \tan^{-1}(\omega) - 180^\circ - \tan^{-1}(\omega/2) - \tan^{-1}(\omega/3)$$

$$\omega=0, \quad M_1 = \infty, \quad \phi_1 = -180^\circ$$

$$\omega=\infty, \quad M_2 = 0, \quad \phi_2 = 90^\circ - 360^\circ = -270^\circ$$

$$E.D \Rightarrow \phi_1 - \phi_2 = -180^\circ + 270^\circ = 90^\circ \text{ (clockwise)}$$

$$S.D \Rightarrow \text{finite zero is near to origin} \\ \text{(anticlockwise)}$$



$$G(s) = \frac{s+3}{s^2(s+1)(s+2)}$$

$$M = \frac{\sqrt{\omega^2 + 9}}{\omega^2 \sqrt{\omega^2 + 1} \sqrt{\omega^2 + 4}}$$

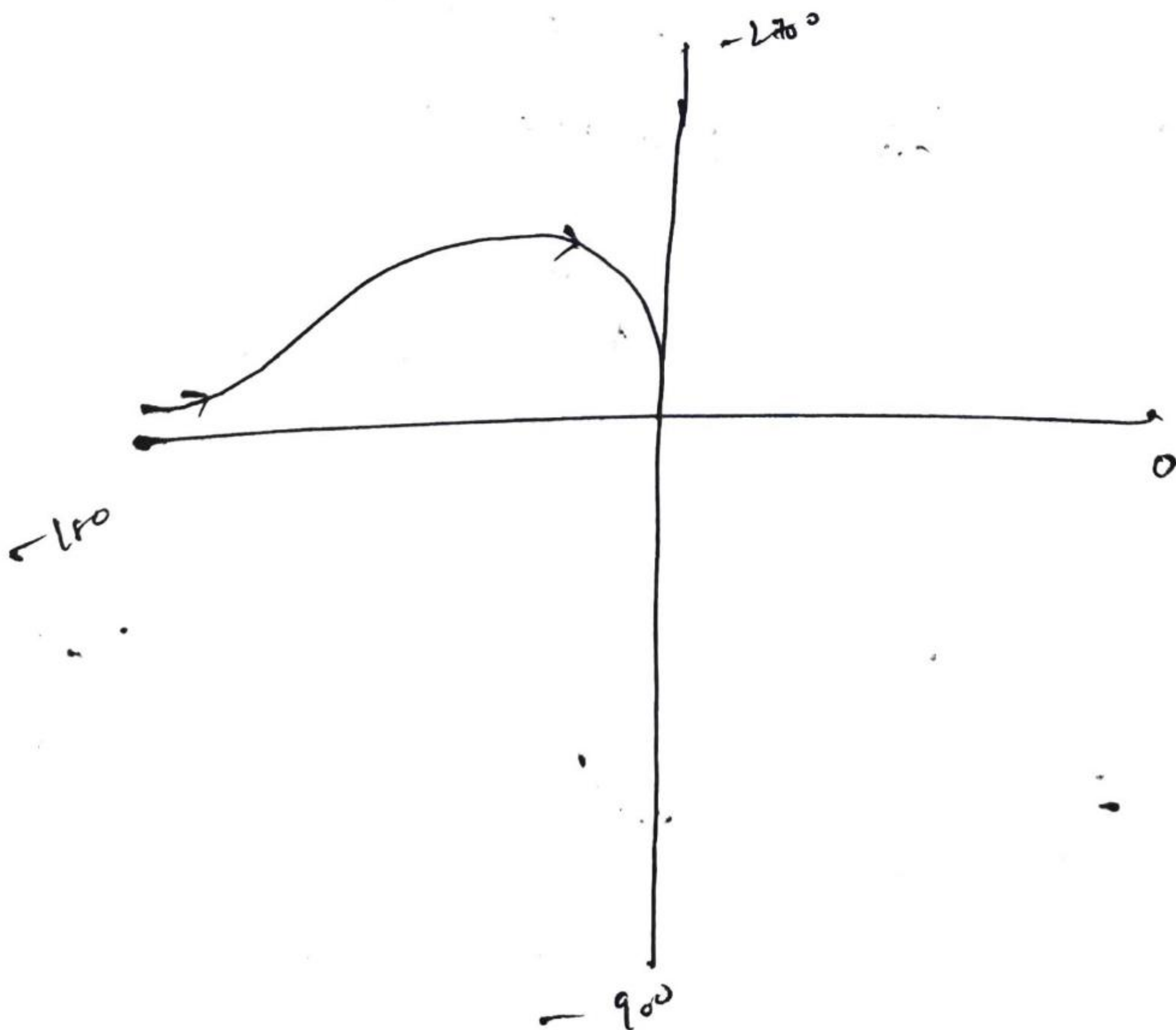
$$\phi = \tan^{-1}(\omega/3) - 180^\circ - \tan^{-1}(\omega) - \tan^{-1}(\omega/2)$$

$$\omega = 0, \quad M_1 = \infty, \quad \phi_1 = -180^\circ$$

$$\omega = \infty, \quad M_2 = 0, \quad \phi_2 = -270^\circ$$

$$\text{E.O} \Rightarrow \phi_1 - \phi_2 = -180^\circ - (-270^\circ) = 90^\circ \text{ (positive)} \\ \text{Clockwise}$$

$$\text{S.D} \Rightarrow \text{finite pole is near to origin} \\ \text{Clockwise}$$



$$G(s) = \frac{(s+2)(s+3)}{s^2(s+1)}$$

$$G(\omega) = \frac{(\omega+2)(\omega+3)}{(\omega^2)(\omega+1)}$$

$$M = \frac{\sqrt{\omega^2+4} \sqrt{\omega^2+9}}{\omega^2 \sqrt{\omega^2+1}}$$

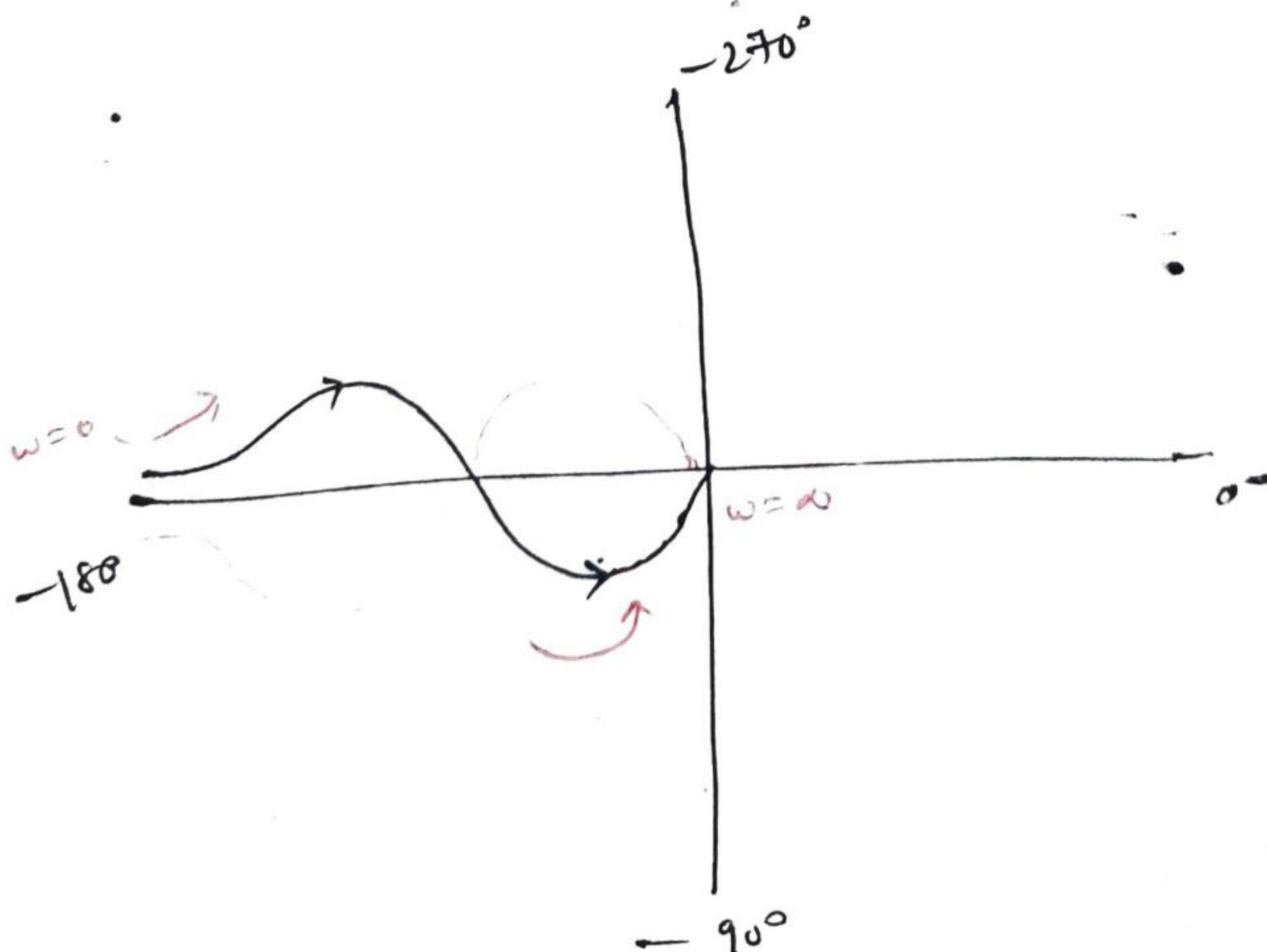
$$\phi = \tan^{-1}(\omega/2) + \tan^{-1}(\omega/3) - 180^\circ - \tan^{-1}(\omega)$$

$$\omega=0, \quad M_1 = \infty, \quad \phi_1 = -180^\circ$$

$$\omega=\infty, \quad M_2 = 0, \quad \phi_2 = -90^\circ$$

$$\text{E.N} \Rightarrow \phi_1 - \phi_2 = -180^\circ + 90^\circ = -90^\circ (A(\omega))$$

S.N $\rightarrow$ Finite pole $\rightarrow$ CW.



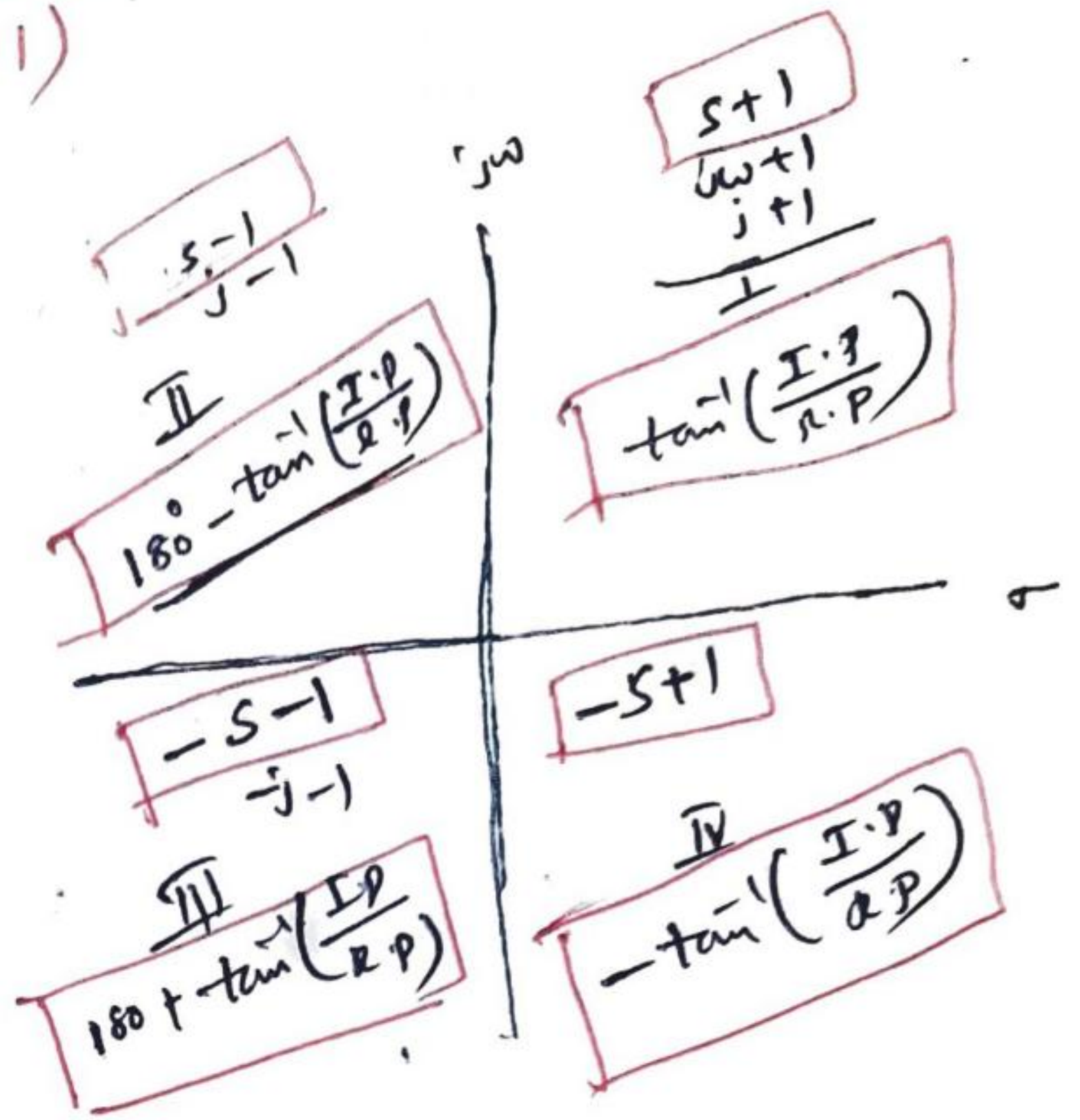
$$G(s) = \frac{1}{s(s-1)}$$

$$G(j\omega) = \frac{1}{j\omega(j\omega-1)}$$

$$M = \frac{1}{\omega \sqrt{\omega^2 + 1}}$$

$$\phi = -90^\circ - [180^\circ - \tan^{-1}(\omega)]$$

$$\phi = -270^\circ + \tan^{-1}(\omega)$$



$$\omega=0, \quad M_1 = \infty$$

$$\phi_1 = -270^\circ$$

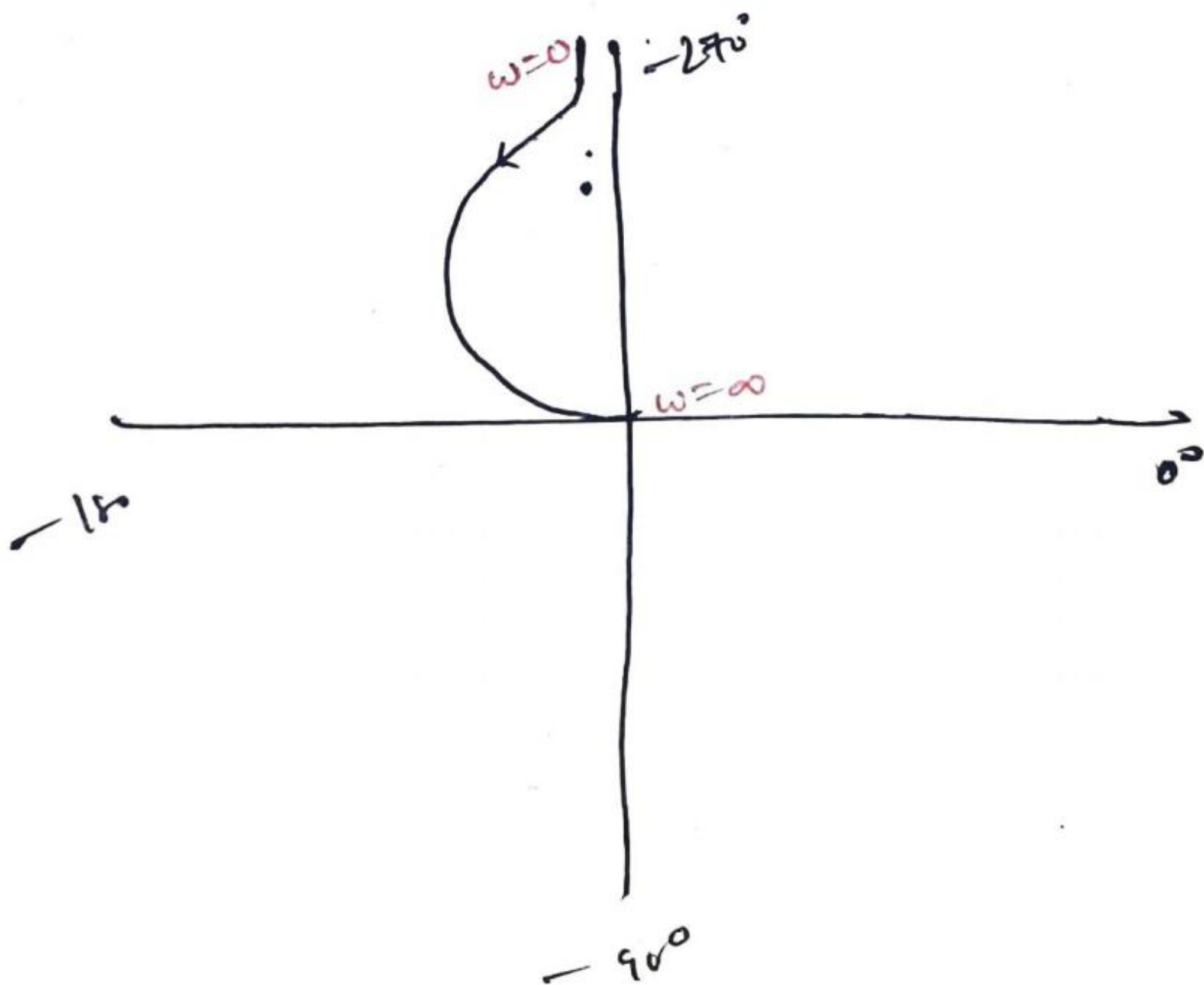
$$\omega=\infty, \quad M_2 = 0$$

$$\phi_2 = -180^\circ$$

$$E.D \Rightarrow \phi_1 - \phi_2 = -270^\circ + 180^\circ = -90^\circ \text{ neg}$$

A.C.W

S.D $\rightarrow$ X (if negative sign is there in T.F)



$$G(s) = \frac{1}{s(-s-1)}$$

$$G(j\omega) = \frac{1}{j\omega(-j\omega-1)}$$

$$M = \frac{1}{\omega \sqrt{\omega^2 + 1}}$$

$$\begin{aligned} \phi &= -90^\circ - [180^\circ + \tan^{-1}(\omega)] \\ &= -270^\circ - \tan^{-1}(\omega) \end{aligned}$$

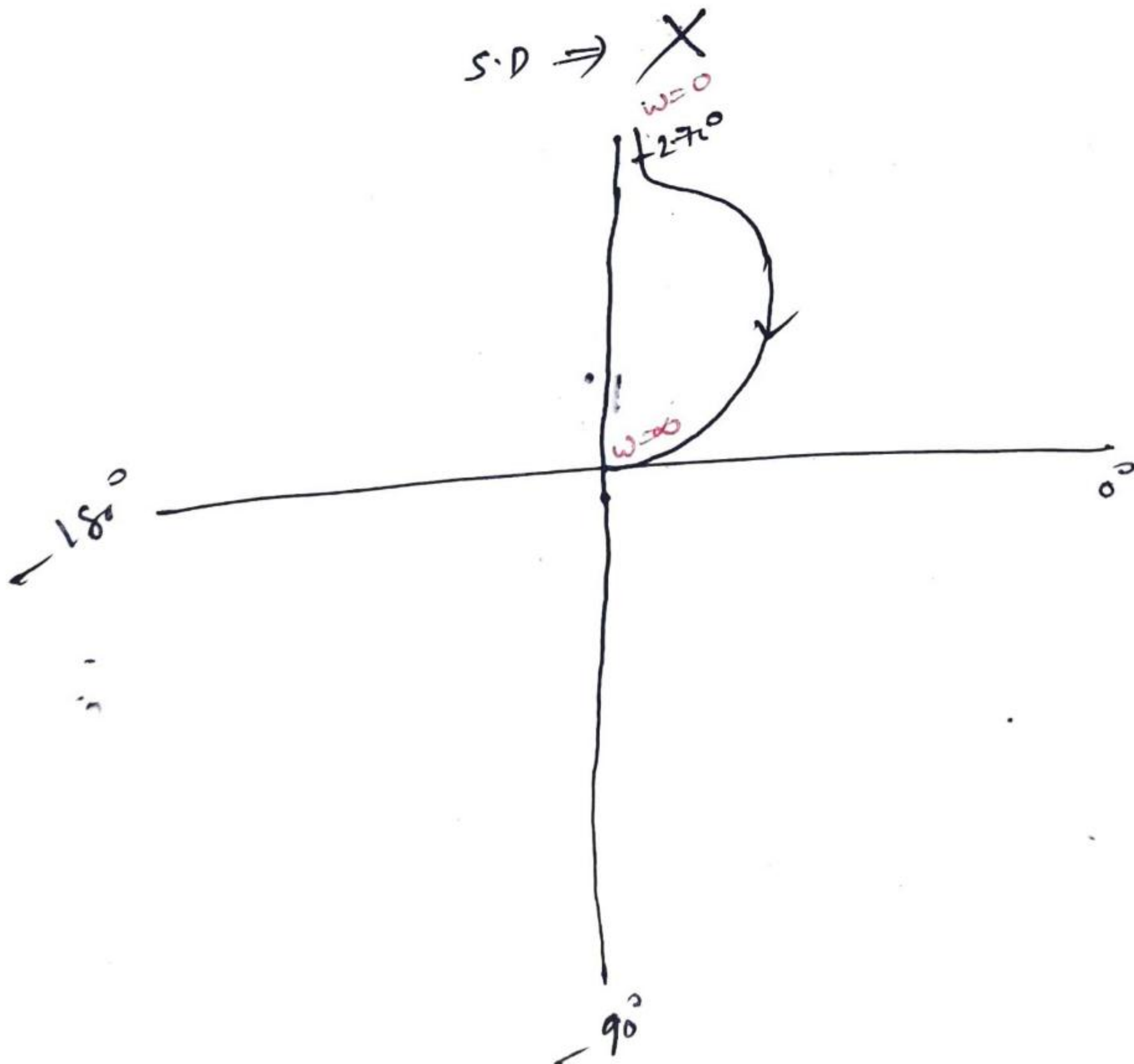
$$\omega = 0, \quad M_1 = \infty, \quad \phi_1 = -270^\circ$$

$$\omega = \infty, \quad M_2 = 0, \quad \phi_2 = -360^\circ$$

$$\text{E.D} \Rightarrow \phi_1 - \phi_2 = +90^\circ \text{ (+ve)}$$

CW.

$$\text{S.D} \Rightarrow \text{X}$$



$$G(s) = \frac{1}{s(1-s)}$$

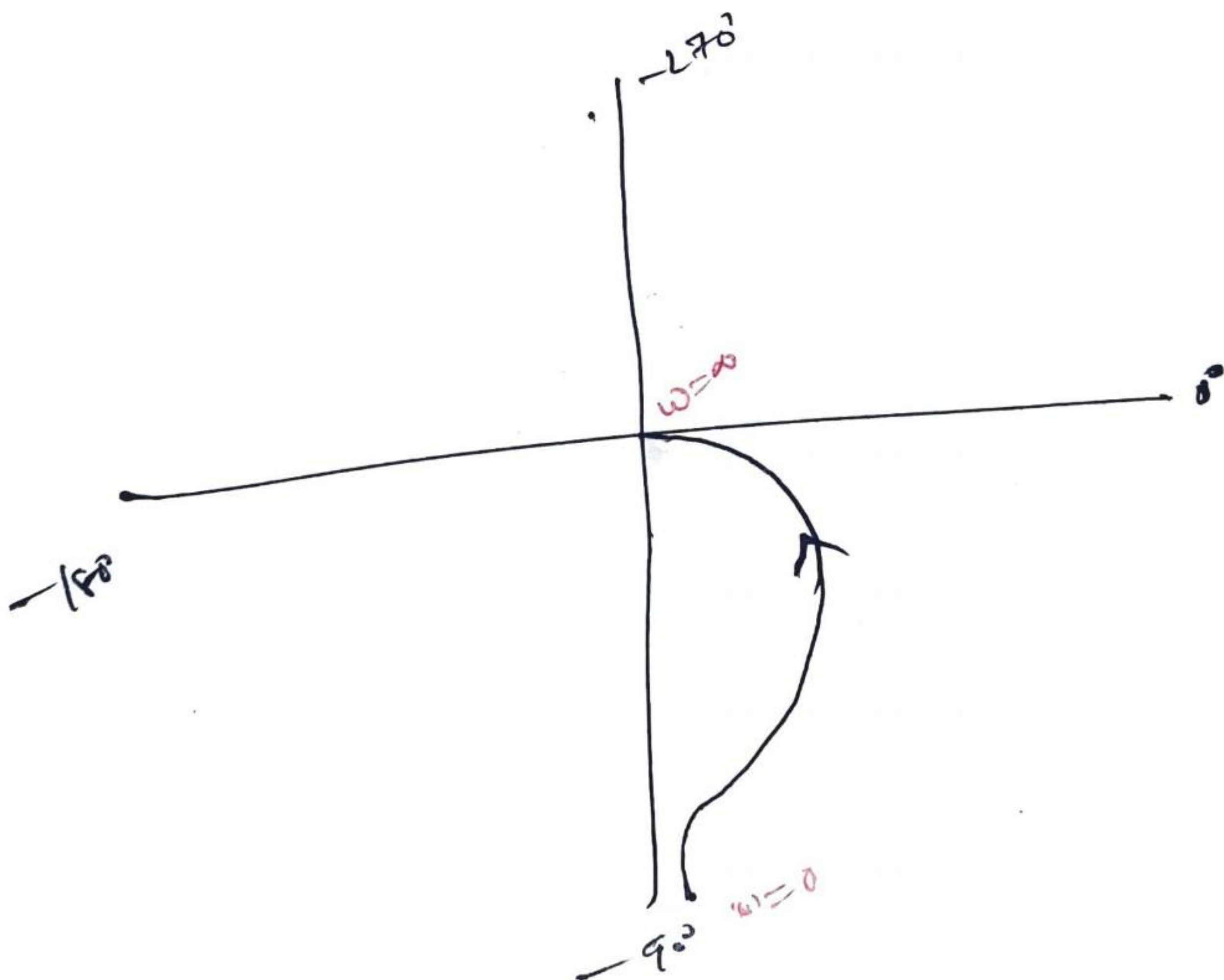
$$M = \frac{1}{\omega \sqrt{1+\omega^2}}$$

$$\begin{aligned}\phi &= -90^\circ - (-\tan^{-1}(\omega)) \\ &= -90^\circ + \tan^{-1}\omega\end{aligned}$$

$$\begin{aligned}\omega=0, \quad M_1 &= \infty, \quad \phi_1 = -90^\circ \\ \omega=\infty, \quad M_2 &= 0, \quad \phi_2 = -90^\circ + 90^\circ = 0^\circ\end{aligned}$$

$$E.D \Rightarrow \phi_1 - \phi_2 = -90^\circ - 0^\circ = -90^\circ$$

$$S.D \Rightarrow X$$



Nyquist plot.

- i) To draw the complete frequency response of OLTT.
- ii) To find no. of ^{CL} poles in RHP of s-plane
- iii) To find Range of 'K' value for stability.
- iv) To find, ω_{gc} , ω_{gc} , GM , PM .
- v) To find Relative stability.

4.4 MATHEMATICAL PRELIMINARIES FOR NYQUIST STABILITY CRITERION

Let $F(s)$ be a function of s , which is expressed as a ratio of two polynomials in s , as shown in equation (4.14), (the polynomials are expressed in the factored form).

$$F(s) = \frac{(s-z_1)(s-z_2) \dots (s-z_m)}{(s-p_1)(s-p_2) \dots (s-p_n)} \quad \dots (4.14)$$

The roots of numerator polynomial are zeros and the roots of denominator polynomial are poles. The function has m number of zeros and n number of poles.

Here, s is a complex variable expressed as, $s = \sigma + j\omega$, where σ is real part of s and ω is imaginary part of s . (The s is also called complex frequency). For a particular value of σ and ω , the s will represent a point in the s -plane.

Since s is a complex variable, the function $F(s)$ will also be a complex quantity for any value of s . Hence, $F(s)$ can also be expressed as, $F(s) = u + jv$, where u is real part of $F(s)$ and v is imaginary part of $F(s)$. Let us define another complex plane called $F(s)$ -plane, with coordinates u and v . For a particular value of s , the $F(s)$ will represent a point in $F(s)$ -plane.

Therefore, for every point s in the s -plane at which $F(s)$ is analytic, there exists a corresponding point $F(s)$ in the $F(s)$ -plane. Hence it can be concluded that the function $F(s)$ maps the points in the s -plane into the $F(s)$ -plane.

Note : A function is analytic in the s -plane provided the function and all its derivatives exist. The points in the s -plane where the function (or its derivatives) does not exist are called singular points.

Since any number of points of analyticity in the s -plane can be mapped into the $F(s)$ -plane it can be concluded that for a contour in the s -plane which does not go through any singular point, there exists a corresponding contour in the $F(s)$ -plane as shown in fig 4.2.

The table 4.2 shows examples of arbitrary s -plane contours and their corresponding $F(s)$ -plane contours (exact shape is not shown).

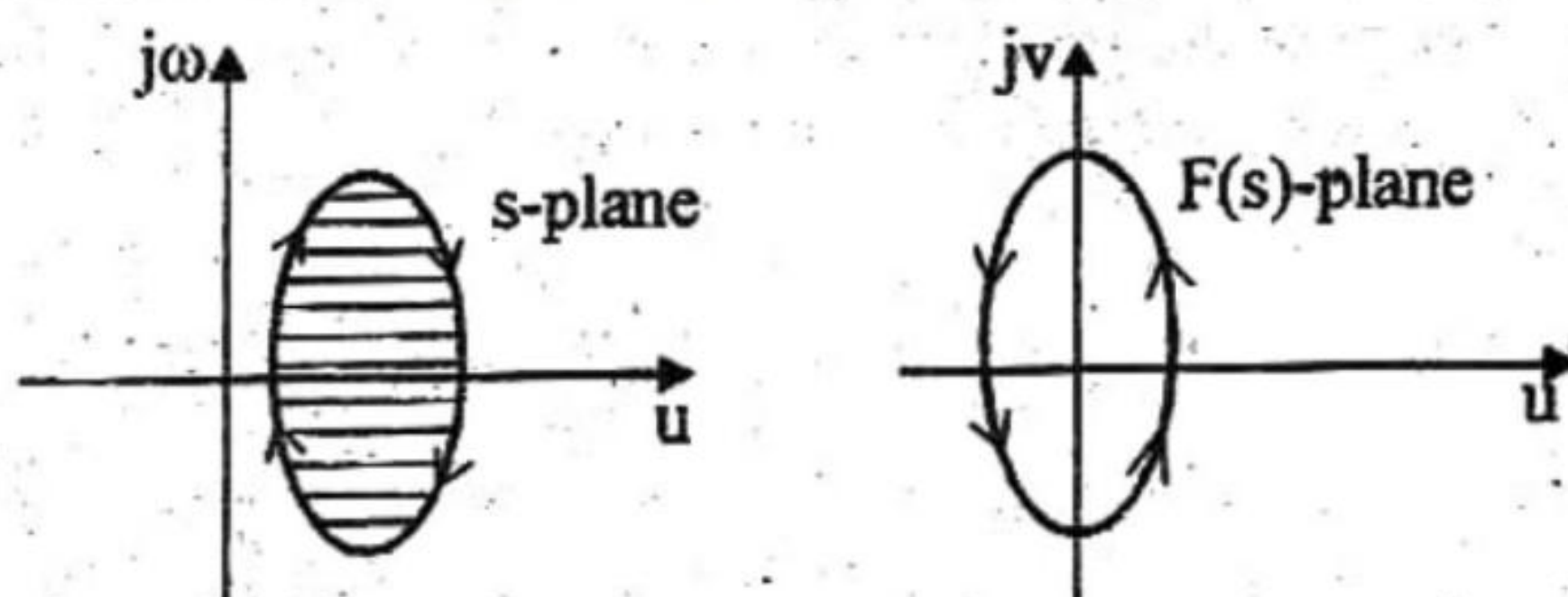


Fig 4.2 : An arbitrary contour in s-plane and its corresponding contour in F(s)-plane

Normally the direction of arbitrary contour in s-plane is chosen as clockwise. Here zeros are marked by small circles (o) and poles by (X).

On observing the s-plane contours and the corresponding F(s)-plane contours shown in table-4.2, it can be proved that there exists a relationship between the enclosure of poles and zeros by the s-plane closed contour and number of encirclements of the origin of F(s)- plane by the corresponding F(s)-plane contour.

Note : For the development of Nyquist criterion, the exact shape of the contour is not required but only the number of encirclements of the origin of the F(s) - plane is essential.

Concept of encircled and enclosed

It is important to distinguish between the concept of encircled and enclosed which are frequently used to apply Nyquist criterion.

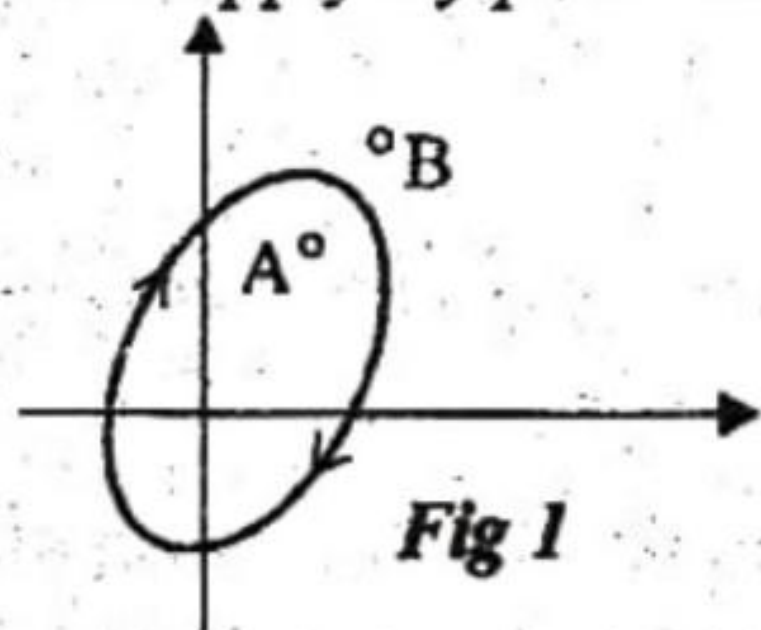


Fig 1

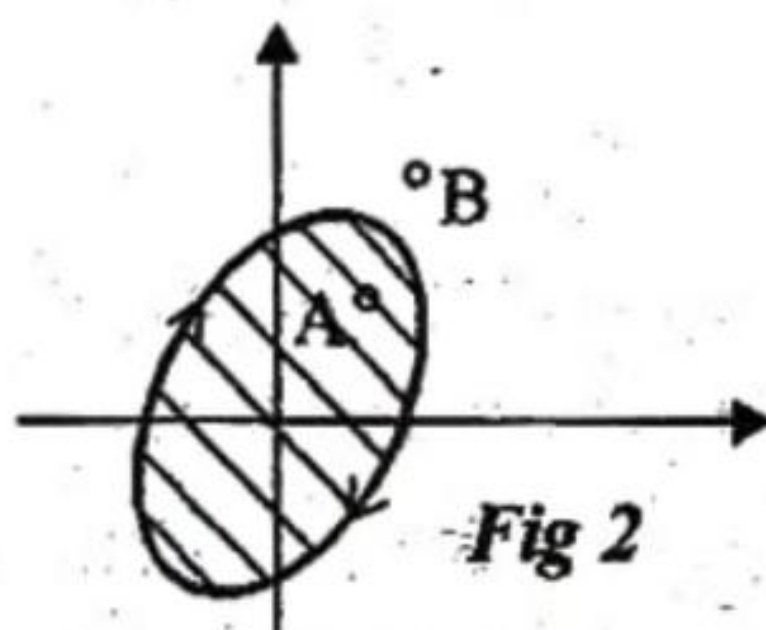


Fig 2

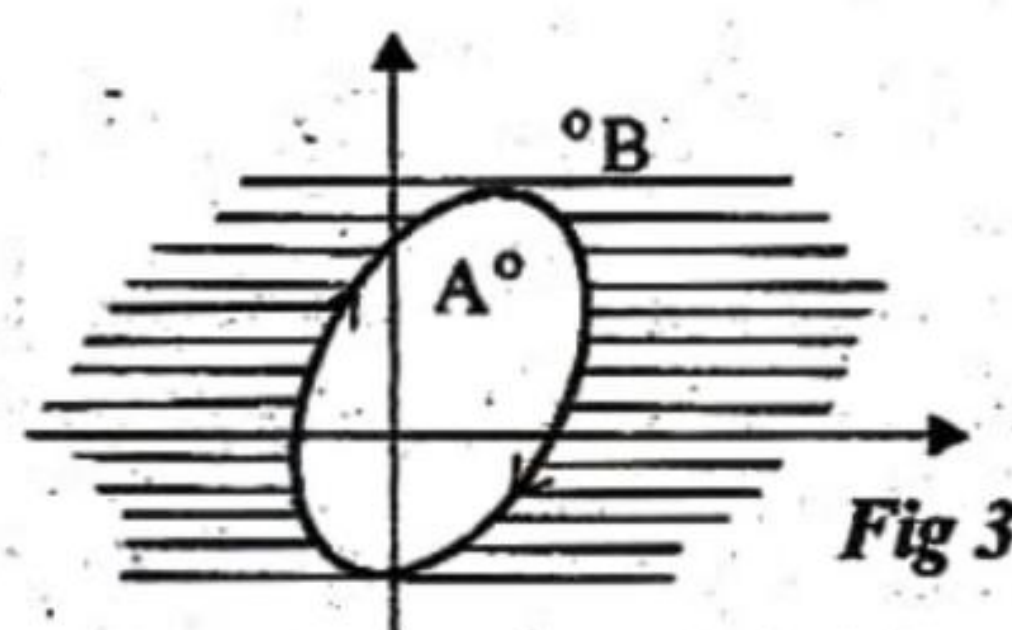


Fig 3

Encircled : A point is said to be encircled by a closed path if it is found inside the path. With reference to fig 1, the point A is encircled in the clockwise direction and the point B is not encircled.

Enclosed : Any point or region is said to be enclosed by a closed path, if it is found to lie to the right of the path when the path is traversed in the prescribed direction. The shaded regions in fig 2 and 3 are the regions enclosed by the closed path. With reference to fig 2, the point A is enclosed by closed path and the point B is not enclosed. With reference to fig 3 the point A is not enclosed by closed path but point B is enclosed.

TABLE-4.2

The function, F(s) and s-plane contour	F(s)-plane contour
$F(s) = s^2 - 2s + 6 = (s - 1 - j2)(s - 1 + j2)$ Zeros : $z_1 = 1 + j2$, $z_2 = 1 - j2$	

$F(s) = \frac{1}{s^2 - 4s + 8} = \frac{1}{(s - 2 - j2)(s - 2 + j2)}$ S-plane Poles : $p_1 = 2 + j2, p_2 = 2 - j2$	F(s)-plane
$F(s) = \frac{s - 1}{(s - 2)(s - 4)}$ s-plane Poles : $p_1 = 2, p_2 = 4$ Zeros : $z_1 = 1$	F(s)-plane
$F(s) = \frac{s^2 - 2s + 6}{s - 2} = \frac{(s - 1 - j2)(s - 1 + j2)}{s - 2}$ s-plane Poles : $p_1 = 2$ Zeros : $z_1 = 1 + j2, z_2 = 1 - j2$	F(s)-plane
$F(s) = \frac{s^2 - 2s + 6}{s^2 - 4s + 8} = \frac{(s - 1 - j2)(s - 1 + j2)}{(s - 2 - j2)(s - 2 + j2)}$ s-plane Poles : $p_1 = 2 + j2, p_2 = 2 - j2$ Zeros : $z_1 = 1 + j2, z_2 = 1 - j2$	F(s)-plane
$F(s) = \frac{s - 2}{(s - 1)(s - 3)}$ s-plane Poles : $p_1 = 1, p_2 = 3$ Zeros : $z_1 = 2$	F(s)-plane

The summary of relationship between the enclosure of poles and zeros by the s -plane contour and number of encirclements of the origin of $F(s)$ -plane by the corresponding $F(s)$ -plane contour, are given below.



1. If s-plane closed contour encloses Z number of zeros in the right half of s-plane then the corresponding contour in $F(s)$ -plane will encircle the origin of $F(s)$ -plane Z times in the clockwise direction.
2. If s-plane closed contour encloses P number of poles in the right half of s-plane then the corresponding contour in $F(s)$ -plane will encircle the origin of $F(s)$ -plane P times in anticlockwise direction.
3. If the s-plane closed contour encloses Z zeros and P poles in the right half of s-plane and if $P > Z$, then the corresponding contour in $F(s)$ -plane will encircle the origin of $F(s)$ -plane $(P - Z)$ times in the anti-clockwise direction.
4. If the s-plane closed contour encloses Z zeros and P poles in the right half of s-plane and if $P < Z$ then the corresponding contour in $F(s)$ -plane will encircle the origin of $F(s)$ -plane $(Z - P)$ times in the clockwise direction.
5. If s-plane closed contour encloses Z zeros and P poles in right half of s-plane and if $P = Z$, then corresponding contour in $F(s)$ -plane will not encircle the origin of $F(s)$ -plane.
6. If the s-plane closed contour does not enclose any pole or zero, then the corresponding contour in $F(s)$ -plane will not encircle the origin of $F(s)$ -plane.

The relation between the enclosure of poles and zeros of $F(s)$ lying on the right half of s-plane by the s-plane contour and the encirclements of the origin of $F(s)$ -plane by the corresponding $F(s)$ -plane contour is called **principle of argument**.

The principle of argument is stated as follows.

Let $F(s)$ is a single valued rational function and is analytic in a given region in the s-plane except at some points. Now, if an arbitrary closed contour is chosen in the s-plane, so that $F(s)$ is analytic at every point on the closed contour in s-plane then the corresponding $F(s)$ -plane contour mapped in the $F(s)$ -plane will encircle the origin N times in anticlockwise direction where N is the difference between the number of poles and number of zeros of $F(s)$ that are encircled by the chosen closed contour in s-plane.

Mathematically, it can be expressed as, $N = P - Z$.

where, N = Number of encirclement of origin of $F(s)$ -plane, made by $F(s)$ -contour.

Z = Number of zeros of $F(s)$ lying on right half of s-plane and enclosed by the s-plane closed contour.

P = Number of poles of $F(s)$ lying on right half of s-plane and enclosed by the s-plane closed contour.

The value of N can be positive, zero or negative. Based on the sign of N , following conclusions can be made, provided the arbitrary s-plane contour is chosen in the clockwise direction.

1. If N is positive, then direction of encirclement of origin of $F(s)$ -plane will be anticlockwise.
2. If N is zero, then there will be no encirclement of origin of $F(s)$ -plane.
3. If N is negative, then direction of encirclement of origin of $F(s)$ -plane will be clockwise.

Principle of argument:

Let $F(s)$ is single valued function and is analytic in a given region in s -plane at some points. If there are 'p' poles and 'z' zeros are enclosed by s -plane closed path, the correspondingly $(G(s) \cdot H(s))$ plane must encircle the origin with $p-z$ times.

Mathematically, it can be expressed as

$$N = P - Z$$

where N = No. of encirclements of origin of $F(s)$ -plane.

Z = No. zeros of $F(s)$ lying on RH of s -plane

P = No. poles of " " " "

- 1) If 'N' is positive, then direction of encirclement of origin of $F(s)$ plane will be anticlockwise
- 2) If 'N' is zero, then there will be no encirclements
- 3) If 'N' is negative, " " " " clockwise.

Nyquist stability criterion:

consider an openloop T.F $G(s) \cdot H(s) = \frac{K N(s)}{D(s)} \rightarrow (1)$

$$\begin{aligned} \text{The CLTF } \frac{C(s)}{R(s)} &= \frac{G(s)}{1 + G(s)H(s)} \\ &= \frac{G(s)}{1 + \frac{K \cdot N(s)}{D(s)}} = \frac{G(s) \cdot D(s)}{D(s) + K N(s)} \rightarrow (2) \end{aligned}$$

The system stability given by C.E

$$\begin{aligned} F(s) &= 1 + G(s) \cdot H(s) \\ &= 1 + \frac{K \cdot N(s)}{D(s)} \\ &= \frac{D(s) + K N(s)}{D(s)} \rightarrow (3) \end{aligned}$$

Now compare (1) and (3)

$$\boxed{\text{poles of C.E (P)} = \text{poles of OLTF}}$$

compare (2) & (3)

$$\boxed{\text{Zeros of C.E (Z)} = \text{poles of CLTF}}$$

$$N = P - Z$$

OLTF poles
lying in RH of
s-plane

poles of CLTF
lying in RH of
s-plane

Open loop system stability

To become an open loop system to be stable there should not be OL pole in the RH of s-plane. The open loop poles in the RH of s-plane nothing but the poles of CE, which must be equal to zero (i.e. $P=0$)

$$N = -Z$$

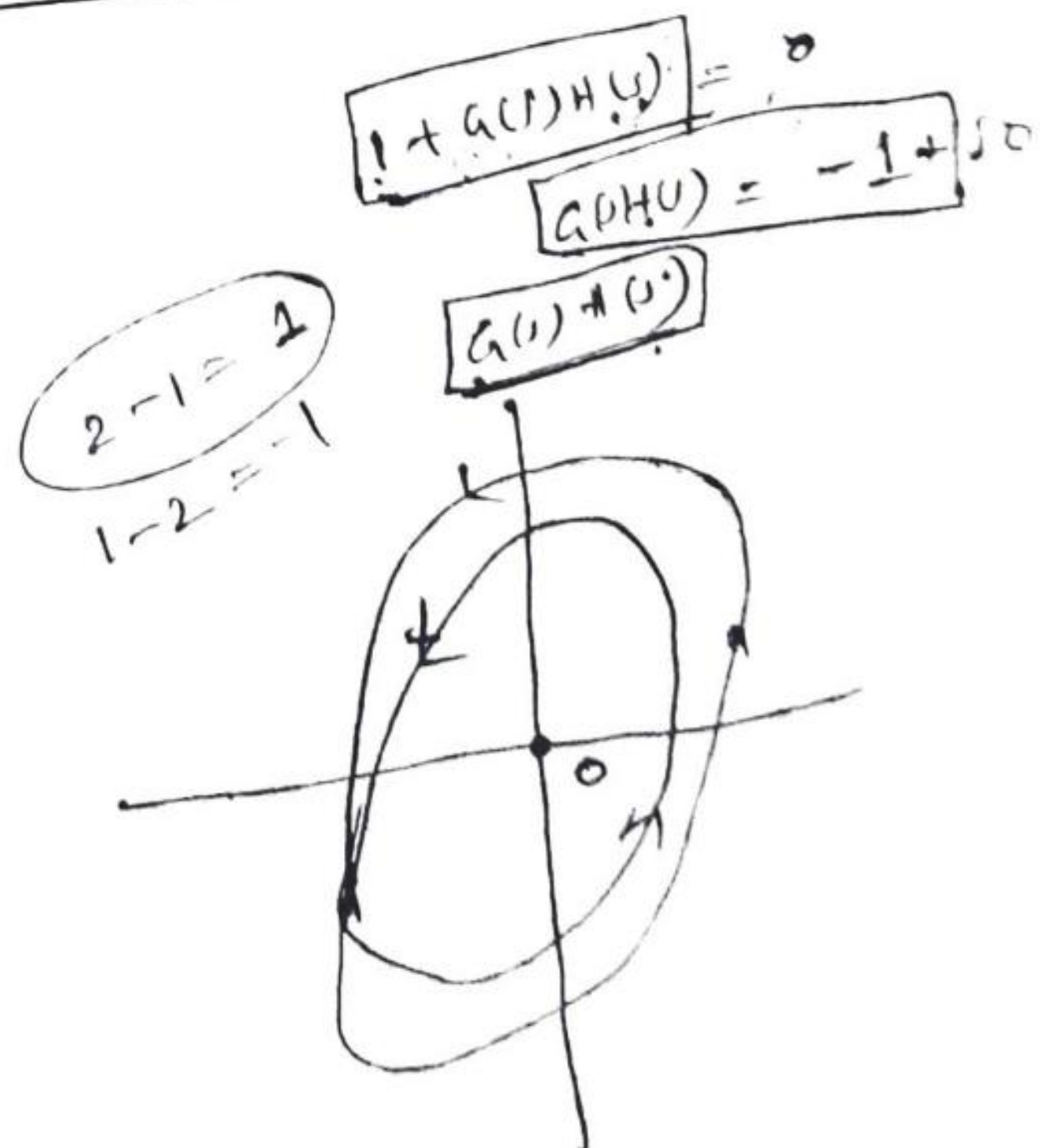
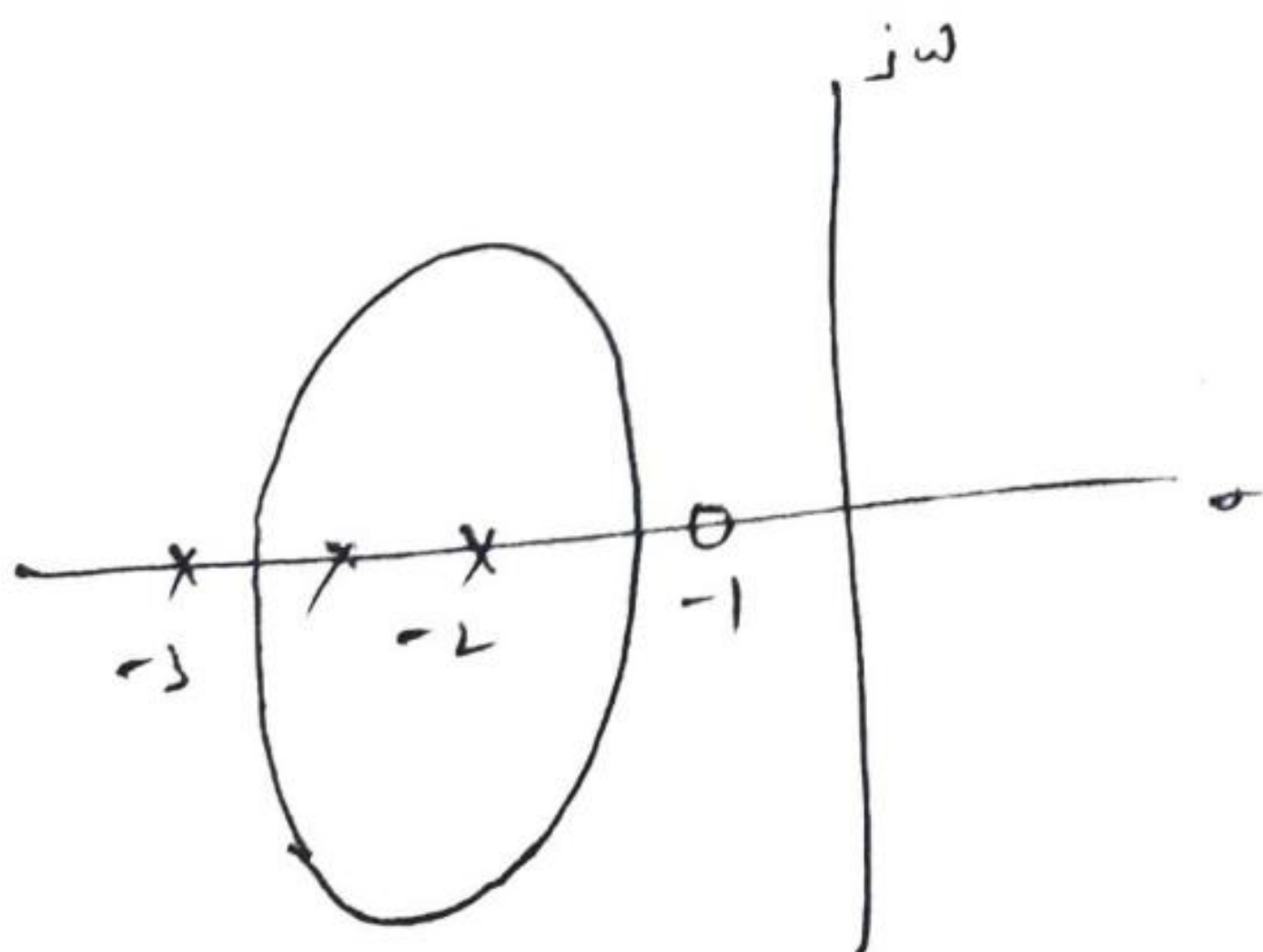
Closed loop system stability

To become closed loop system to be stable there should not be any CL pole lying on R-H of s-plane i.e. $P(Z=0)$.

$$N = P$$

Eg:

$$G(s)H(s) = \frac{s+1}{(s+2)(s+3)}$$



Nyquist stability criterion can be stated as follows.

If the $G(s)H(s)$ contour in the $G(s)H(s)$ plane corresponding to the Nyquist contour in the s -plane encircles the point $-1+j0$ in anti-clockwise direction as many times as the number of right half s -plane poles of $G(s)H(s)$, then the CL system is stable.

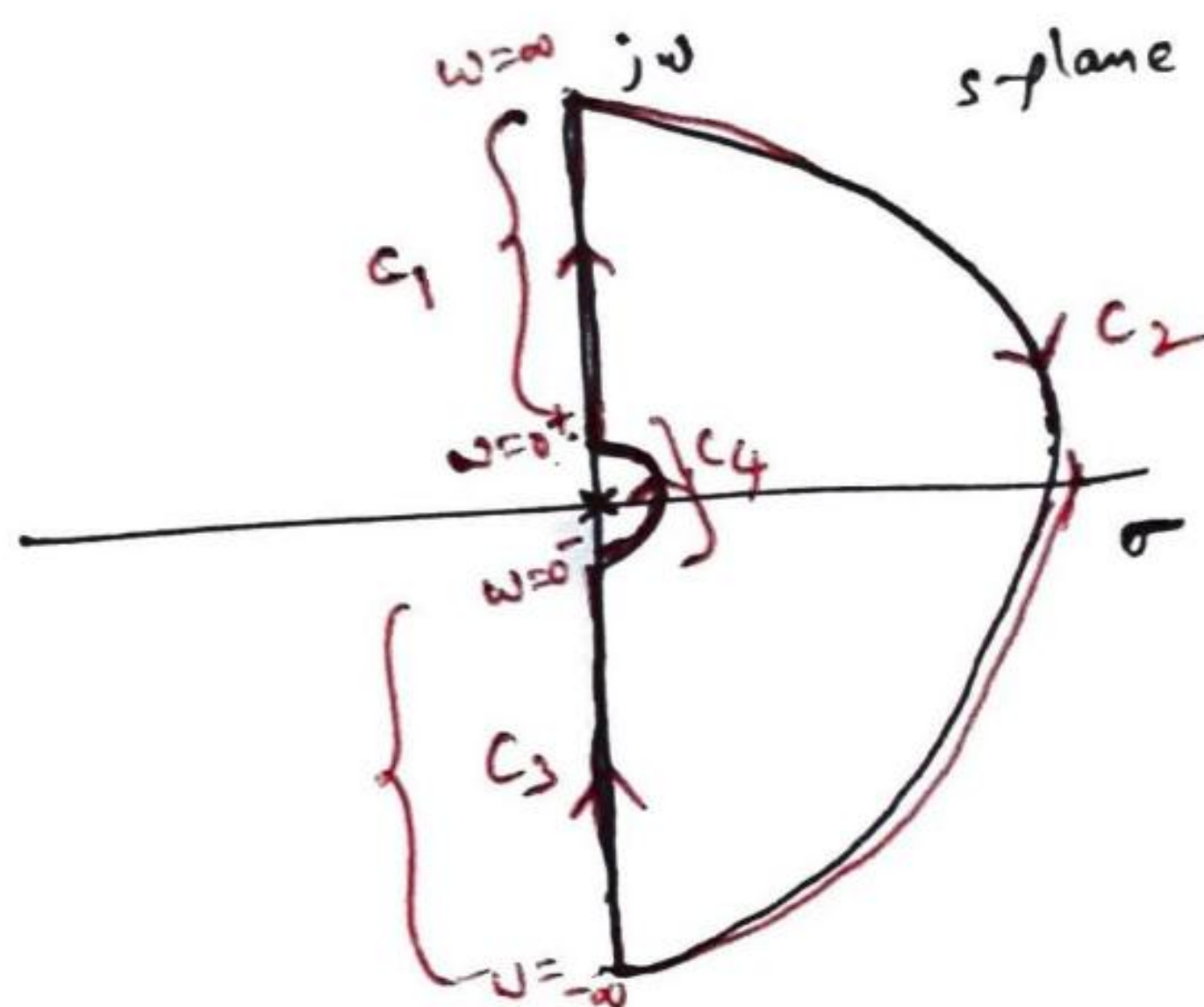
$$N = P$$

steps to plot Nyquist plot:-

Consider Nyquist contour when there are poles at origin.

procedure to draw Nyquist plot:

consider Nyquist contour when there are poles at origin.



→ In section C_1 , ω varies from 0 to $+\infty$. The mapping of section C_1 is given by locus of $G(s)H(s)$ as ω is varied from 0 to ∞ . This locus is the polar plot of $G(j\omega)H(j\omega)$.

→ The mapping of section C_2 from s-plane $G(s)H(s)$ plane is obtained by letting $s = \lim_{R \rightarrow \infty} R \cdot e^{j\theta}$ in $G(s)H(s)$ and varying θ from $+\pi/2$ to $-\pi/2$. Since $s \rightarrow R e^{j\theta}$ and $R \rightarrow \infty$.

then $G(s)H(s)$ can be approximated by.

No. of poles at origin

$\Rightarrow$ No. of infinite radius half circles.

Eg:

$$G(s)H(s) = \frac{K}{s^2}$$

$$s = \lim_{R \rightarrow \infty} R \cdot e^{j\theta}$$

$$\Rightarrow G(s)H(s) = \frac{K}{\lim_{R \rightarrow \infty} [R \cdot e^{j\theta}]^2}$$

$$= \infty e^{-j2\theta}$$

$$\text{when } \theta = \pi/2 \Rightarrow G(s)H(s) = \infty e^{-j/2 \cdot \pi} \Rightarrow \pi$$

$$\theta = -\pi/2 \Rightarrow G(s)H(s) = \infty e^{+j/2 \cdot \pi} \Rightarrow -\pi$$

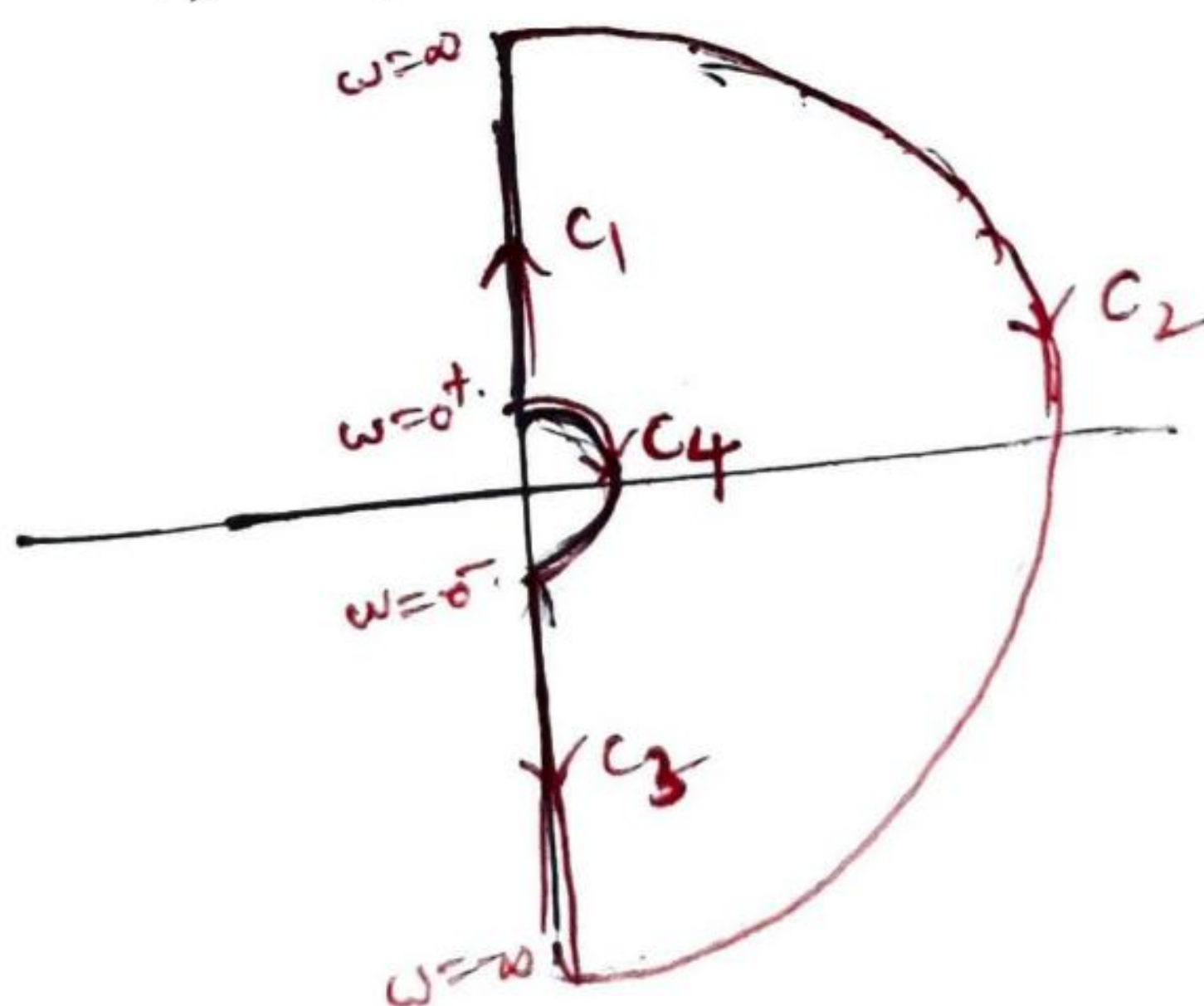
→ In section C_1 , ω varies from $-\infty$ to 0 . This locus is the inverse polar plot of $G(j\omega) \cdot H(j\omega)$, [i.e mirror image of C_1 w.r.t. Real axis].

→ In the section C_2 , ω is varies from $-\infty$ to ∞ . It represents origin in $G(s) \cdot H(s)$ plane.

when $R \rightarrow \infty$, $M \rightarrow 0$.

Sketch the Nyquist plot for $G(s)H(s) = \frac{1}{s(s+1)}$

Sol: Consider Nyquist contour in s -plane



The contour has 4 sections C_1, C_2, C_3, C_4 . The mapping of each section is performed separately and overall nyquist plot obtained by combining the individual sections.

Mappig of section 'C₁' :

In section 'C₁' 'ω' varies from 0 to ∞.

The mappig of section 'C₁' is noting but polar plot of given OLTF.

Given OLTF : $G(s)H(s) = \frac{1}{s(s+1)}$

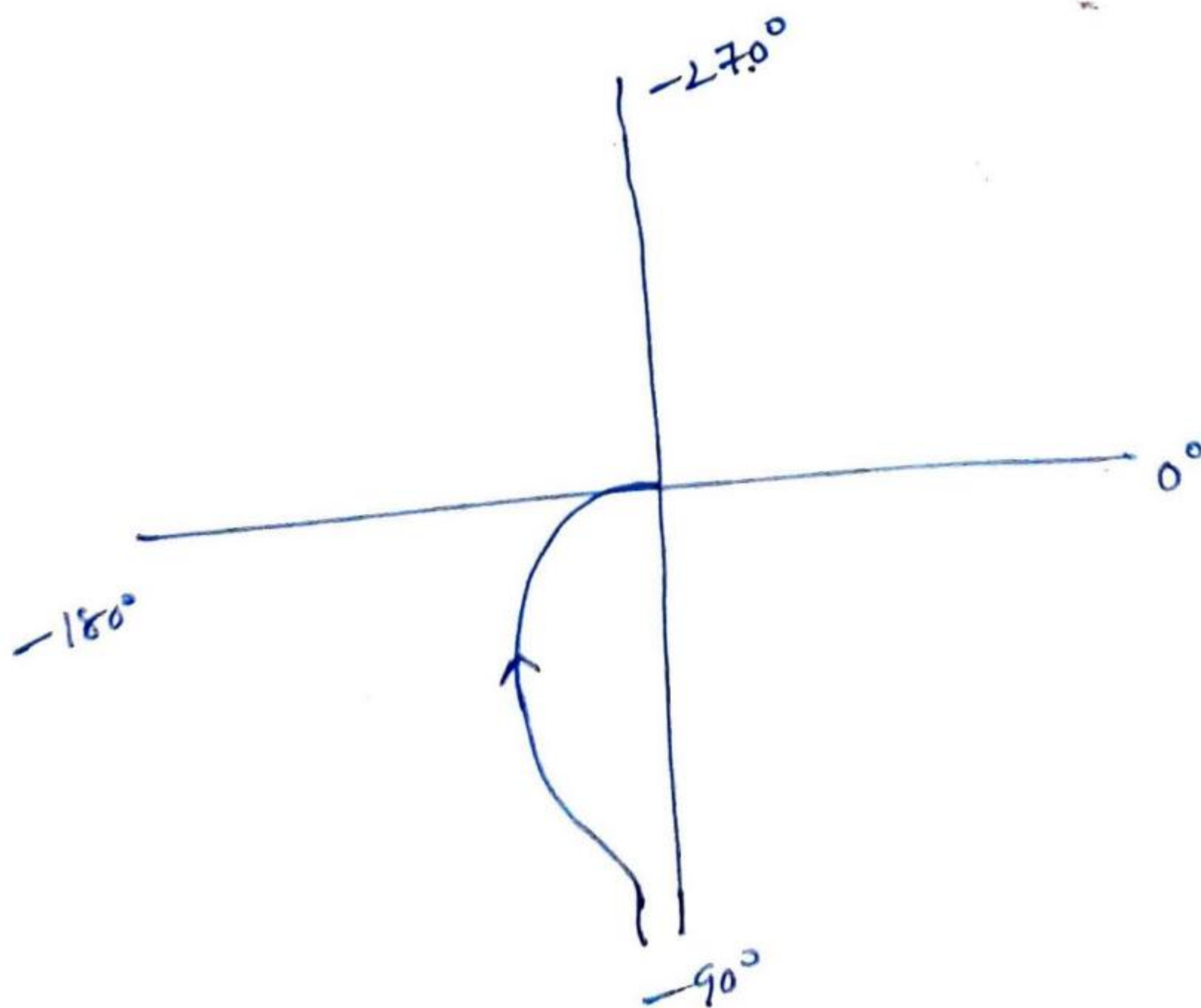
$$G(j\omega)H(j\omega) = \frac{1}{j\omega(j\omega+1)}$$

$$M = \frac{1}{\omega \sqrt{\omega^2+1}}, \quad \theta = -90^\circ - \tan^{-1} \omega.$$

when $\omega = 0$, $M_1 = \infty$, $\theta_1 = -90^\circ$

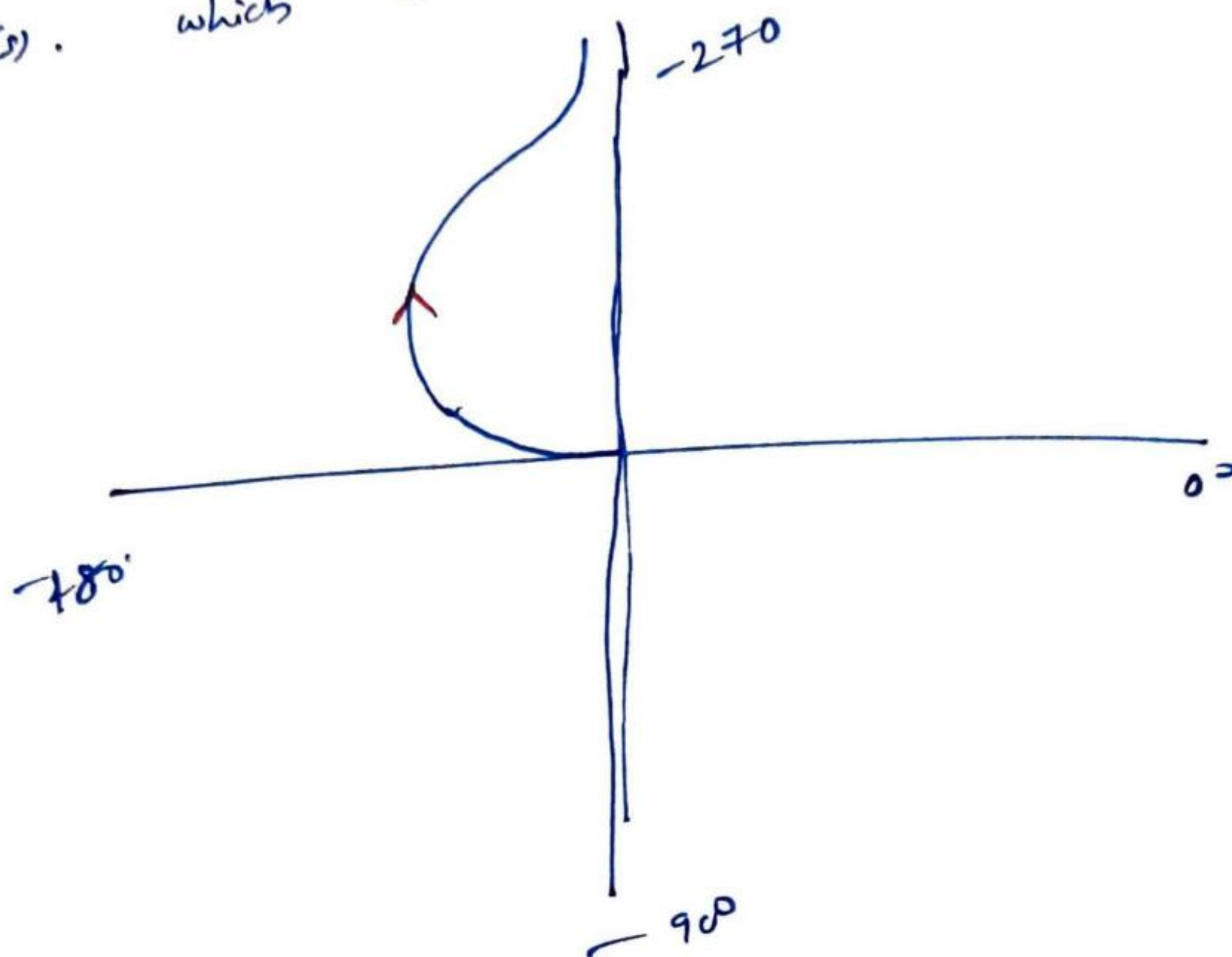
$\omega = \infty$, $M_2 = 0$, $\theta_2 = -180^\circ$

E.D $\Rightarrow \theta_1 - \theta_2 = -90^\circ + 180^\circ = +90^\circ$
CW.



Mapping of section C_3

In section C_3 ω varies from $-\infty$ to 0 i.e. mapping of section C_3 is nothing but inverse polar plot of $G(s) \cdot H(s)$ which is mirror image of C_1 .



Mapping of section C_2

In section C_2 ω varies from $-\infty$ to ∞ . We can conclude that the section C_2 of Nyquist plot is mapped as origin. (Zero).

Mapping of section C_4

In section C_4 ω is varies from 0^- to 0^+ i.e. $s = \lim_{R \rightarrow 0} R \cdot e^{j\theta}$.

The section C_4 of Nyquist contour has a semicircle of infinite radius.

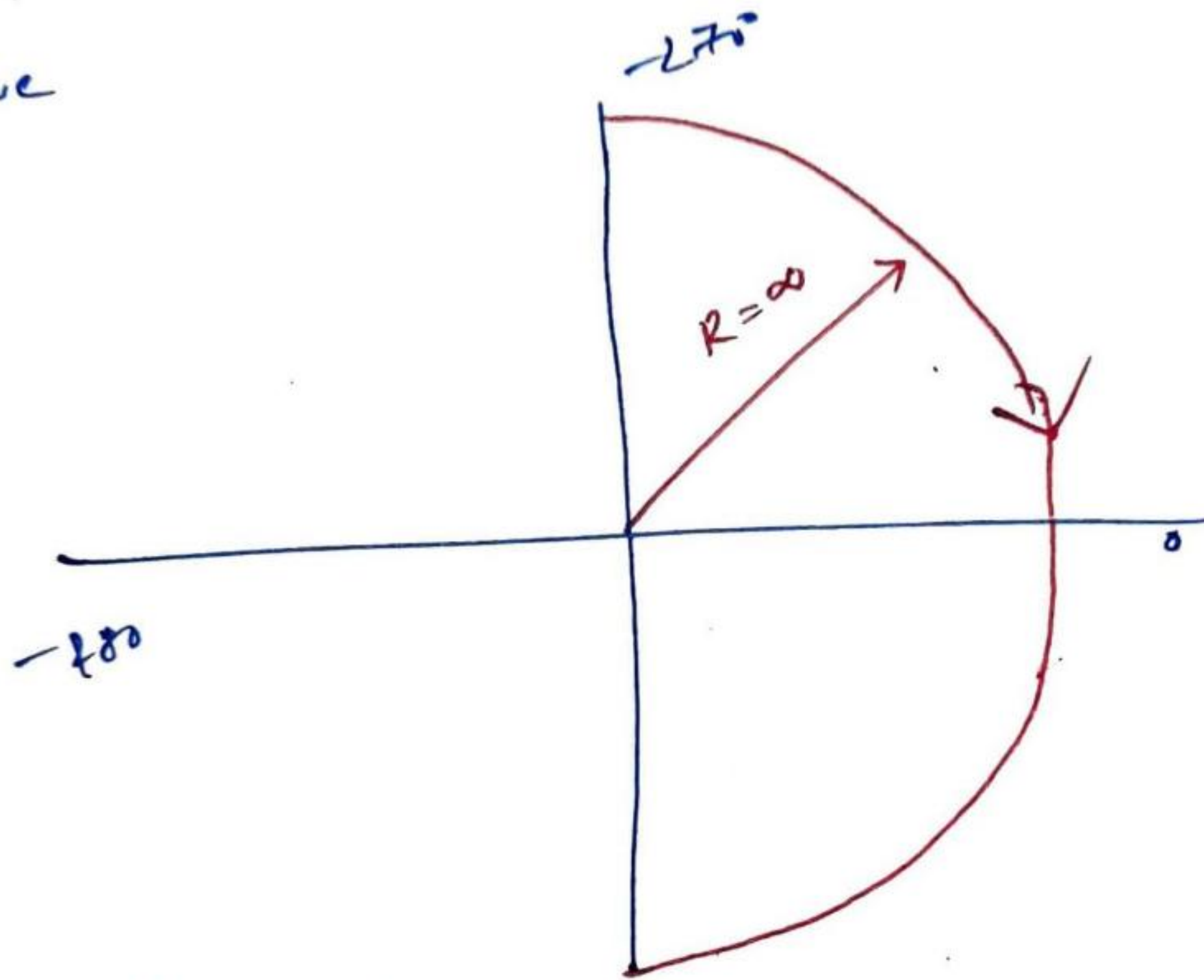
No. of semi circles = No. of poles at origin

In given T.F, we have one pole at origin.

and we will get one half circle in the mapping of C_4 .

The infinite radius half circle ~~start~~ should start at where mirror image starts and ends at where actual poles plot starts.

clockwise



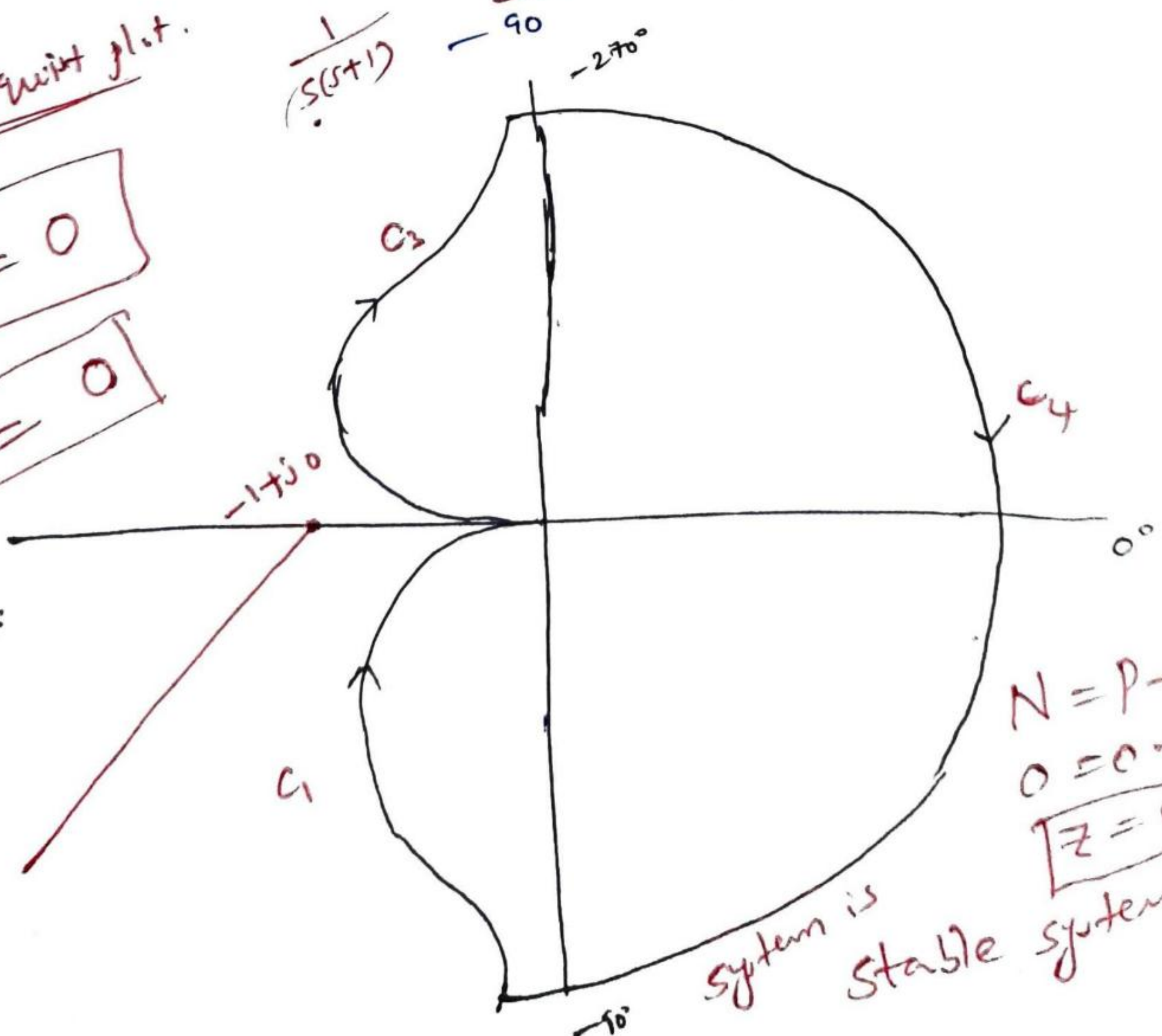
Complete Nyquist plot.

$$N = 0$$

$$P = 0$$

No. of
OL poles in
R.H. s-plane
-1/2

$$\frac{1}{s(s+1)}$$



$$N = P - Z$$

$$0 = 0 - Z$$

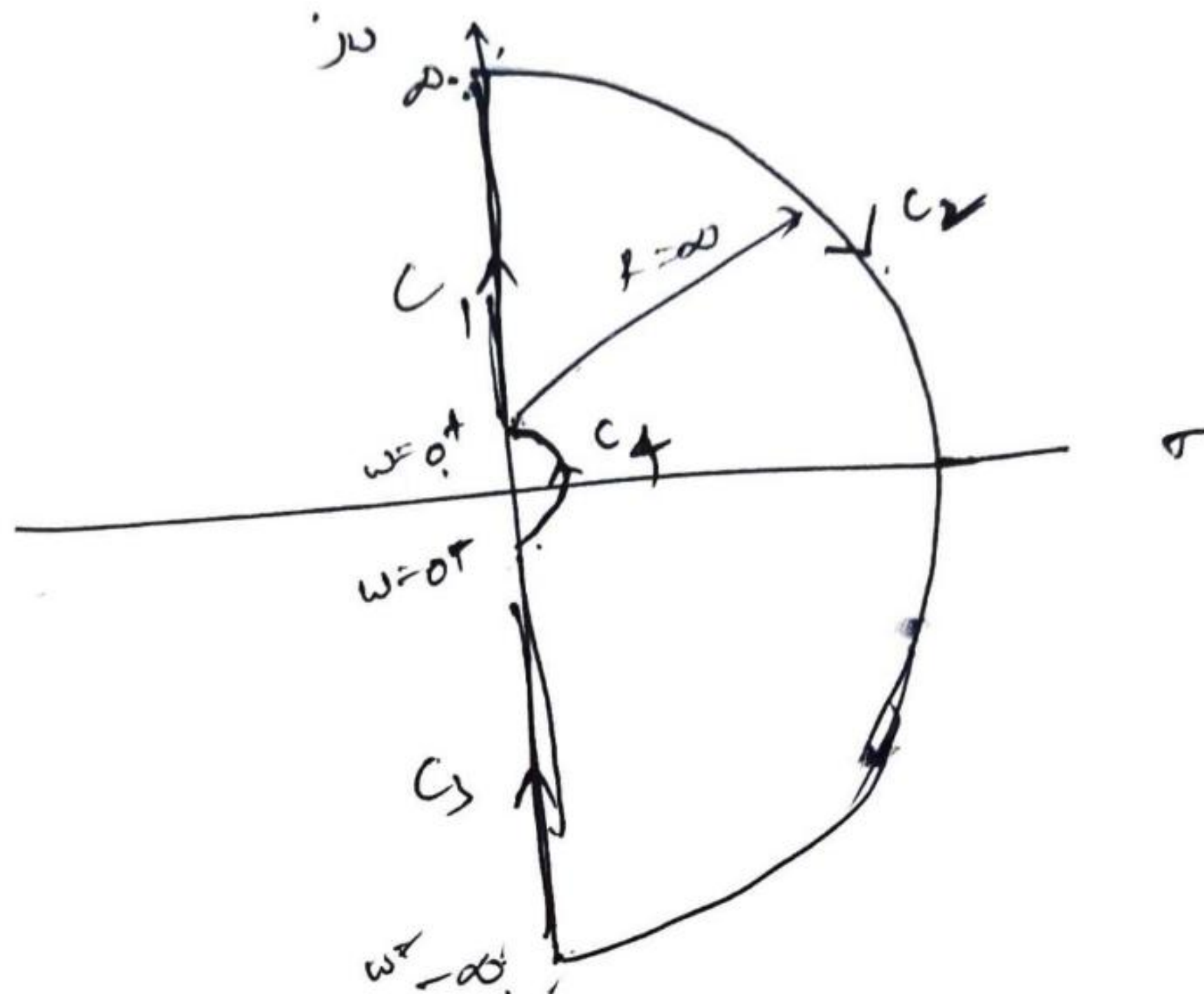
$$Z = 0$$

system is stable systems

Draw the Nyquist plot and determine the range of 'K' for which the closed loop system is stable.

$$G(s)H(s) = \frac{K}{s(s+2)(s+10)}$$

Sol.



Section C1

$$G(s)H(s) = \frac{K}{s(s+2)(s+10)}$$

$$G(j\omega)H(j\omega) = \frac{K}{j\omega(j\omega+2)(j\omega+10)}$$

$$M = \frac{K}{\omega \sqrt{\omega^2+4} \sqrt{\omega^2+100}}$$

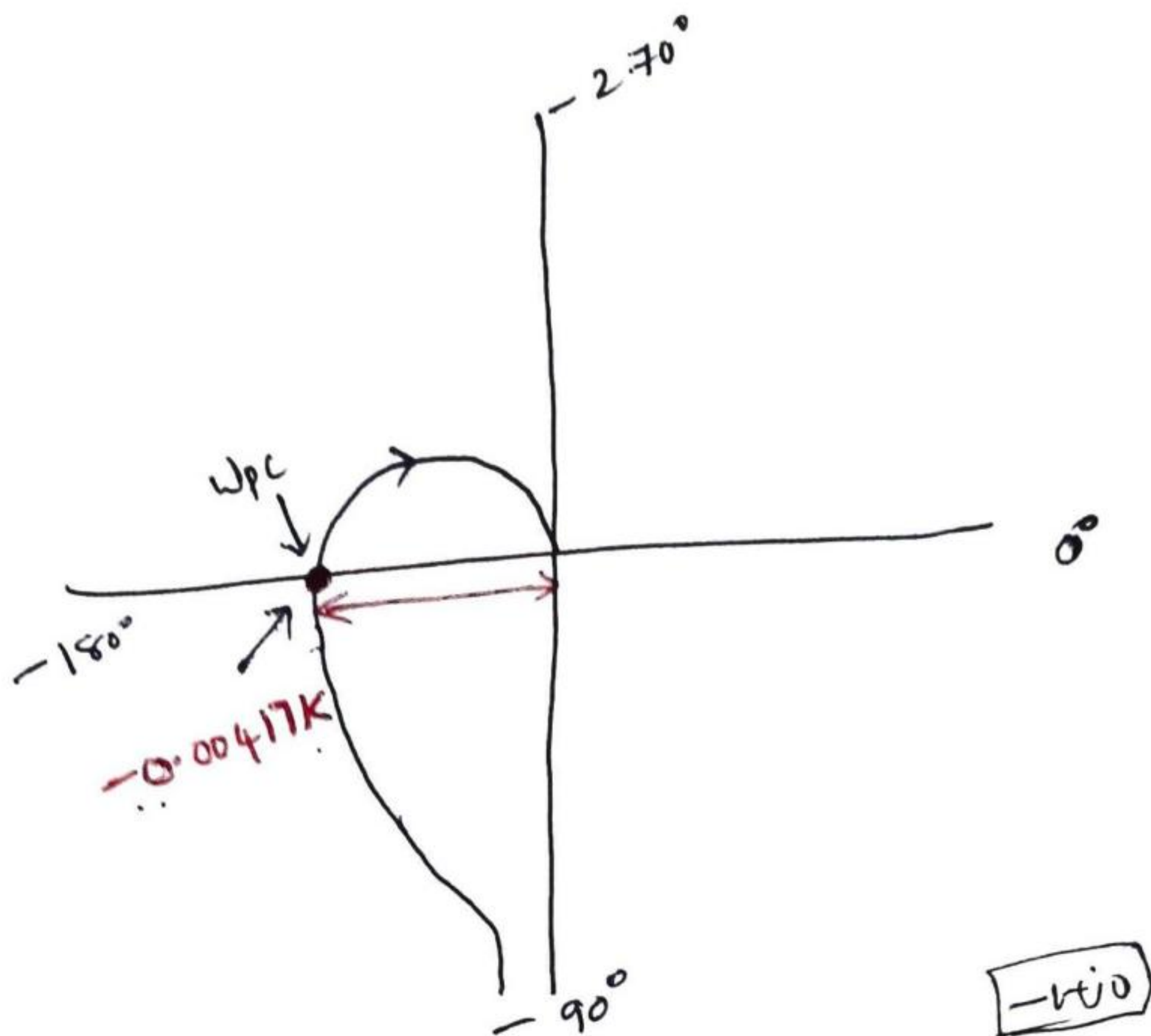
$$\theta = -90^\circ - \tan^{-1}(\omega/2) - \tan^{-1}(\omega/10)$$

When $\omega=0$, $M_1 = \infty$, $\theta_1 = -90^\circ$

$\omega=\infty$, $M_2 = 0$, $\theta_2 = -270^\circ$

E.D $\Rightarrow \theta_1 - \theta_2 = -90^\circ + 270^\circ = +180^\circ$

Clockwise direction



When the locus of $G(j\omega)H(j\omega)$ crosses real axis, then imaginary term will be zero. and the corresponding frequency is cross over frequency (ω_{pc}) .

$$G(s)H(s) = \frac{K}{s(s+2)(s+10)}$$

$$= \frac{K}{s(s^2+12s+20)} = \frac{K}{s^3+12s^2+20s}$$

$$\underline{s = j\omega}$$

consider denominator

$$(j\omega)^3 + 12(j\omega)^2 + 20(j\omega)$$

Equate Imag. terms to zero

$$(j\omega_{pc})^3 + 20(j\omega_{pc}) = 0$$

$$\omega_{pc} [-\omega_{pc}^2 + 20] = 0$$

$$\omega_{pc}^2 = 20 \Rightarrow \omega_{pc} = 4.47 \text{ rad/sec}$$

Magnitude at ω_{pc} $M = \frac{K}{\omega \sqrt{\omega^2+4} \sqrt{\omega^2+10}} \Big|_{\omega=\omega_{pc}}$

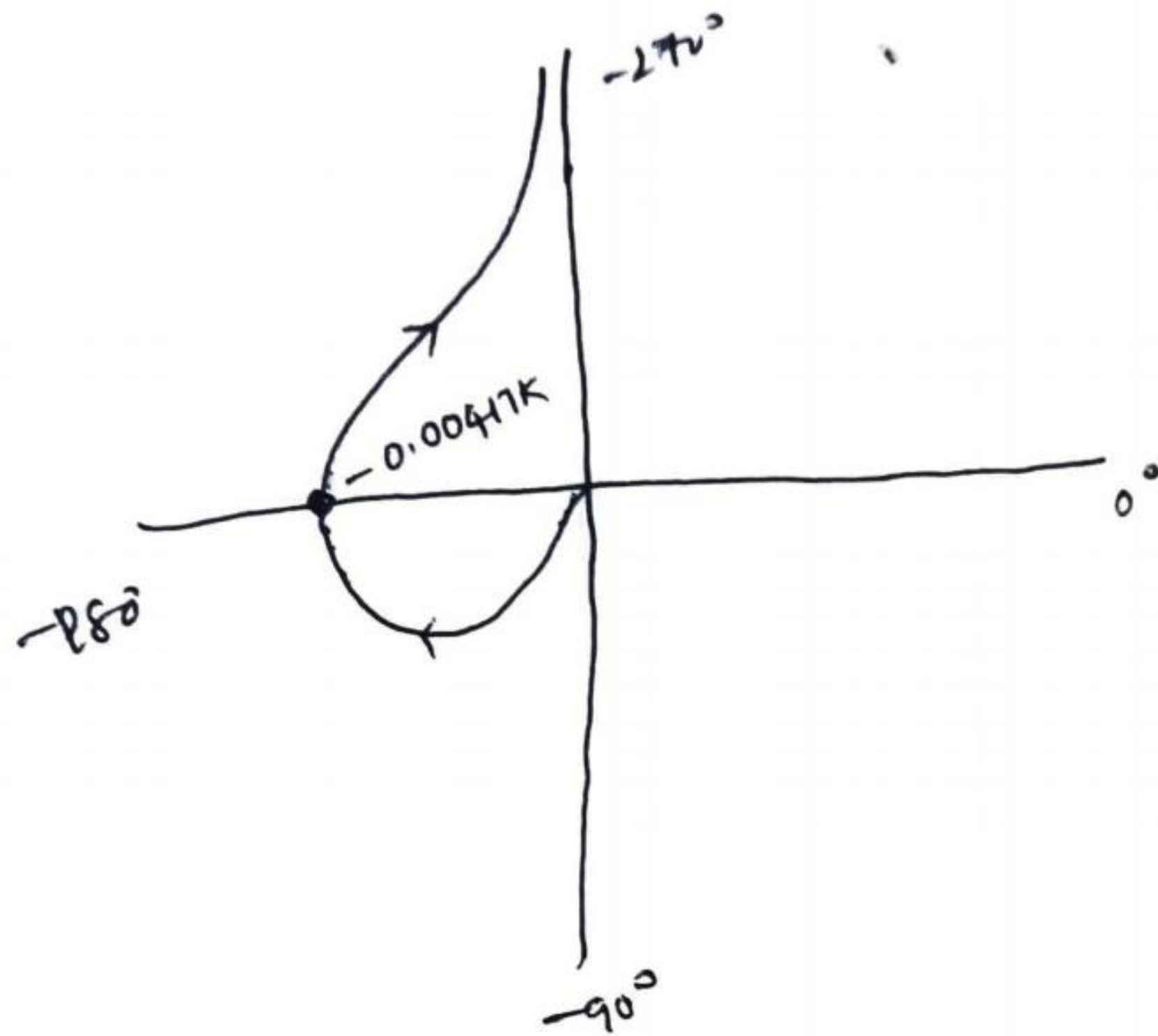
$$= -0.00417 K$$

$$R \rightarrow \infty, M \rightarrow 0$$

Section C₂

It is always zero

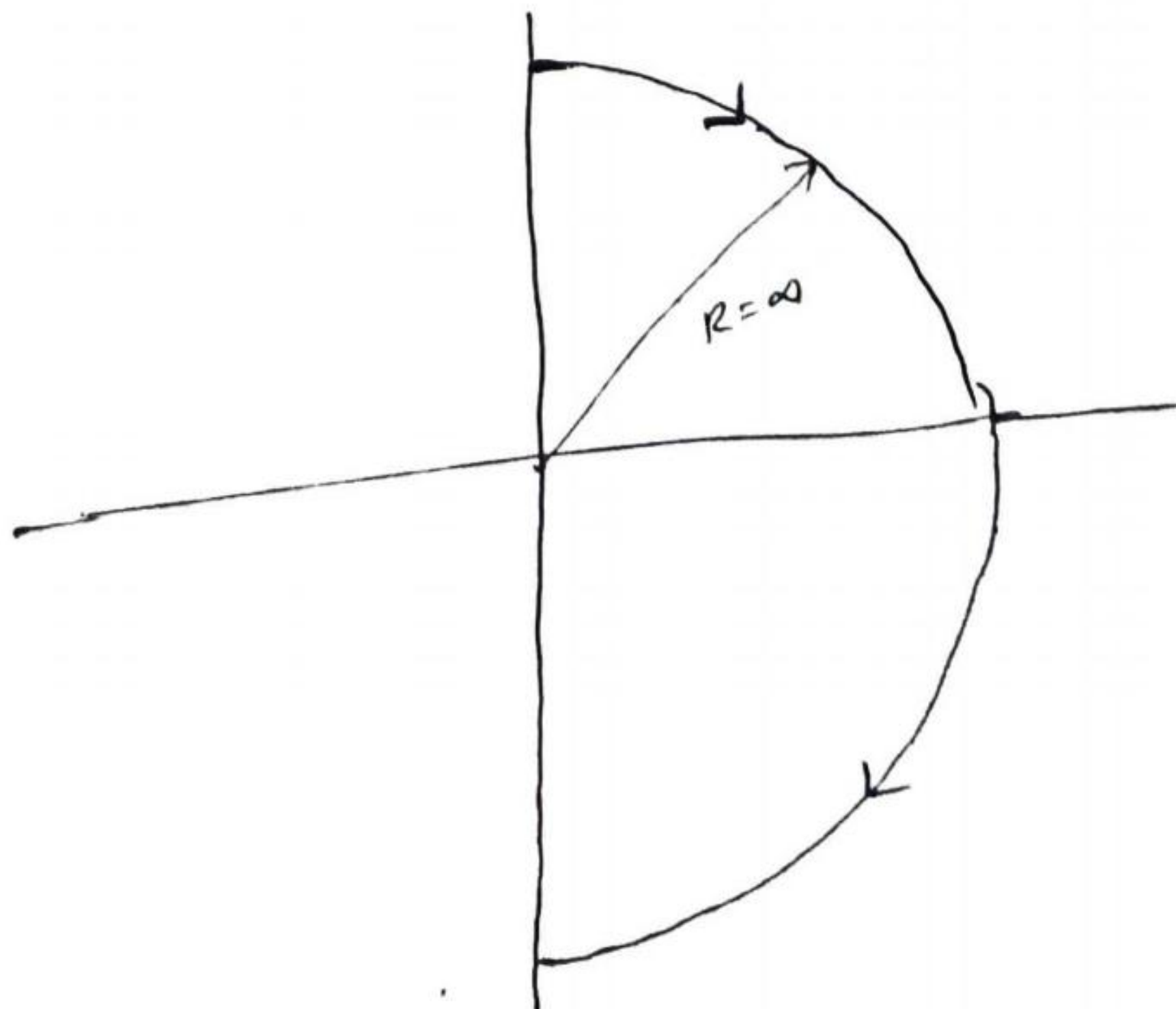
Section C₃ :- Inverse polar plot of $G(s)H(s)$ i.e mirror image of section C₁ w.r.t. real axis.



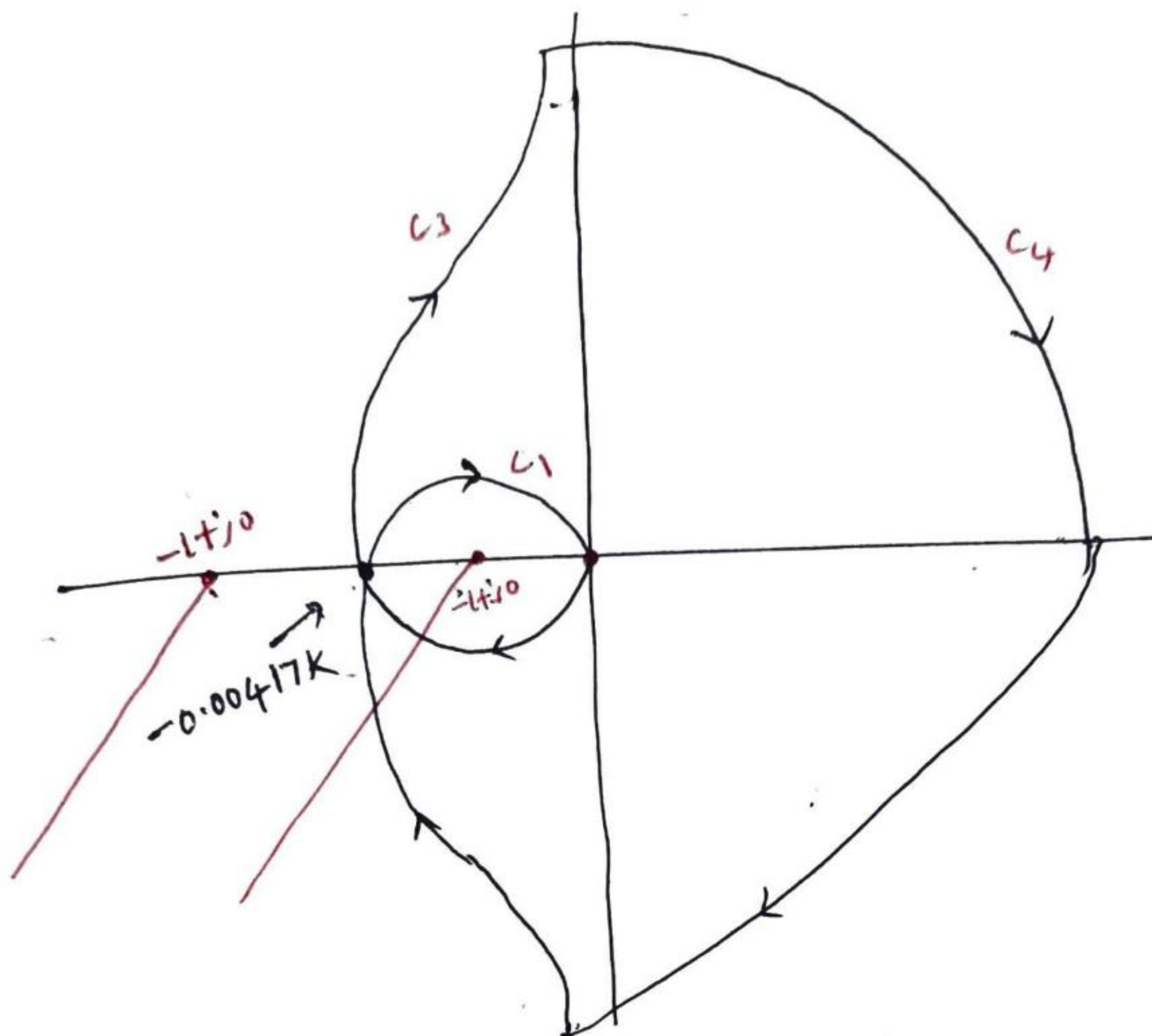
Section C₄

No. of poles at origin = 1

C₄ consists of one half circle of infinite radius.



complete Nyquist plot.



Given OLTF $G(s)H(s) = \frac{K}{s(s+2)(s+10)}$

OLTF poles lying on RH s-plane

$$P = 0$$

$$N = P - Z$$

$$N = 0 - Z \Rightarrow \boxed{N = -Z} \rightarrow (1)$$

To CL system to be stable Z should be zero.
i.e. from (1), we should have $\boxed{N = 0}$ (No encirclements)

If $+0.00417K < 1$, then only $N = 0$

$$K < \frac{1}{0.00417} = 240$$

$$\boxed{K < 240}$$

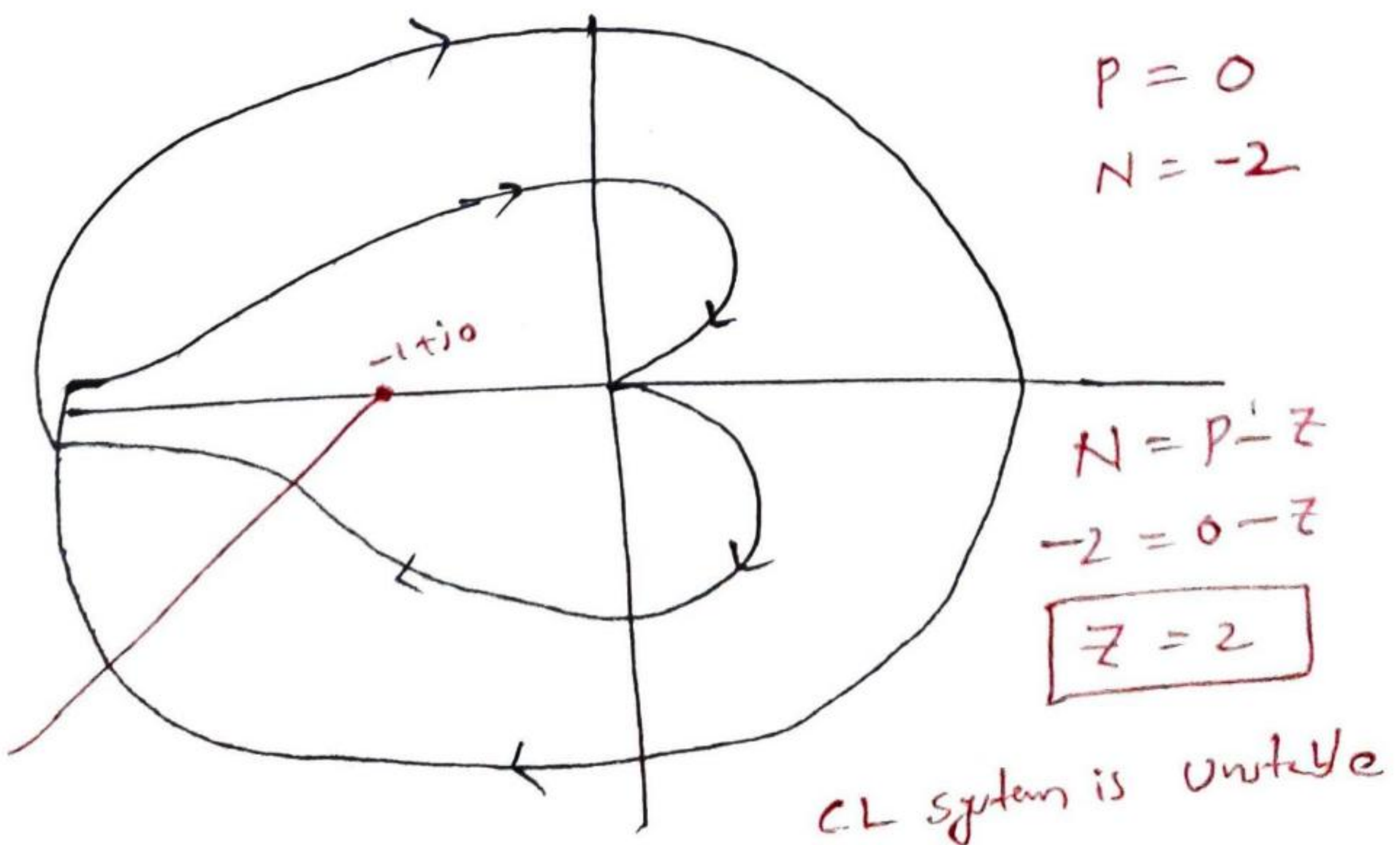
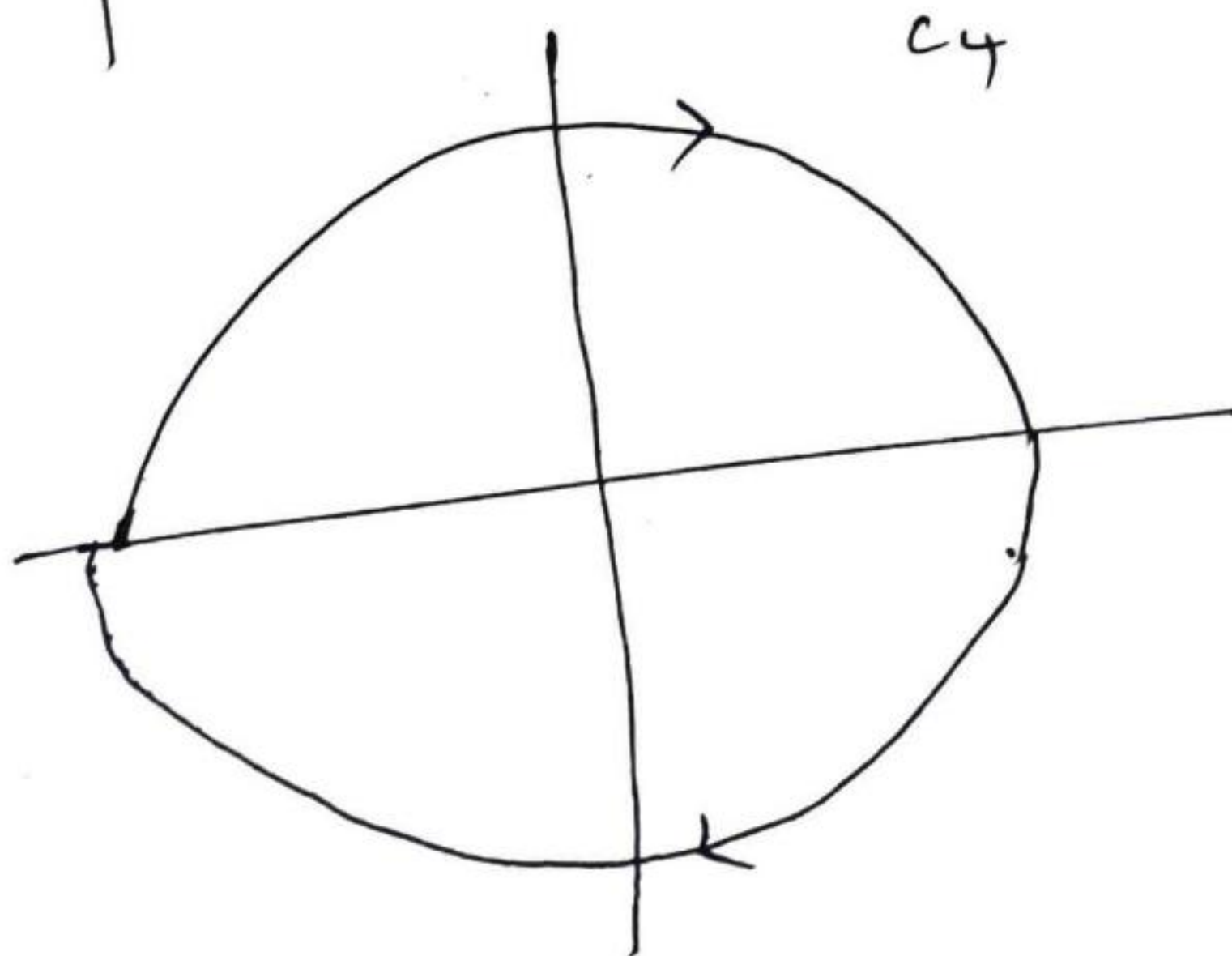
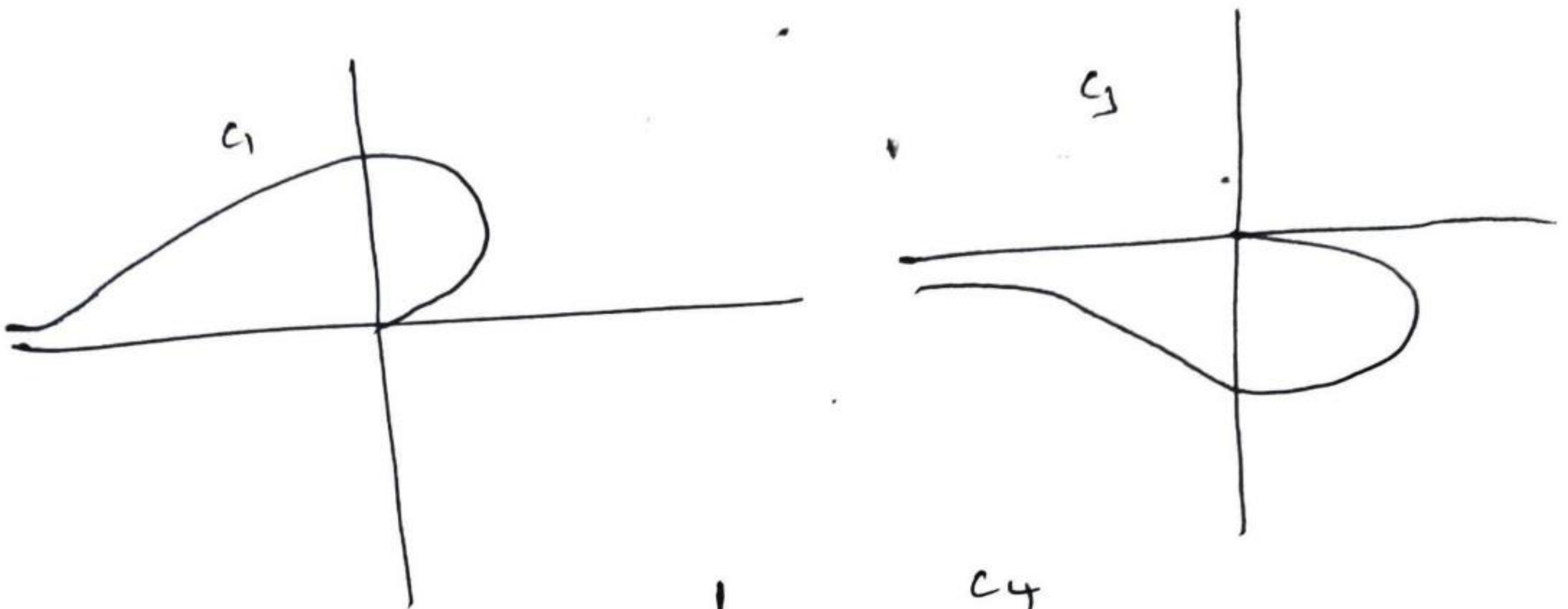
$$\boxed{0 < K < 240}$$

→

$$G(s)H(s) = \frac{10}{s^2(s+4)(s+5)}$$

$$\omega=0, \quad M_1 = \infty, \quad \theta_1 = -180^\circ$$

$$\omega=\infty, \quad M_\infty = 0, \quad \theta_\infty = -360^\circ$$



$$G(s)H(s) = \frac{K(1+s)^2}{s^3}$$

Find Nyquist plot and Range of k for CL system to be stable.

Sol.

Section C1

$$G(s)H(s) = \frac{K(1+s)^2}{s^3}$$

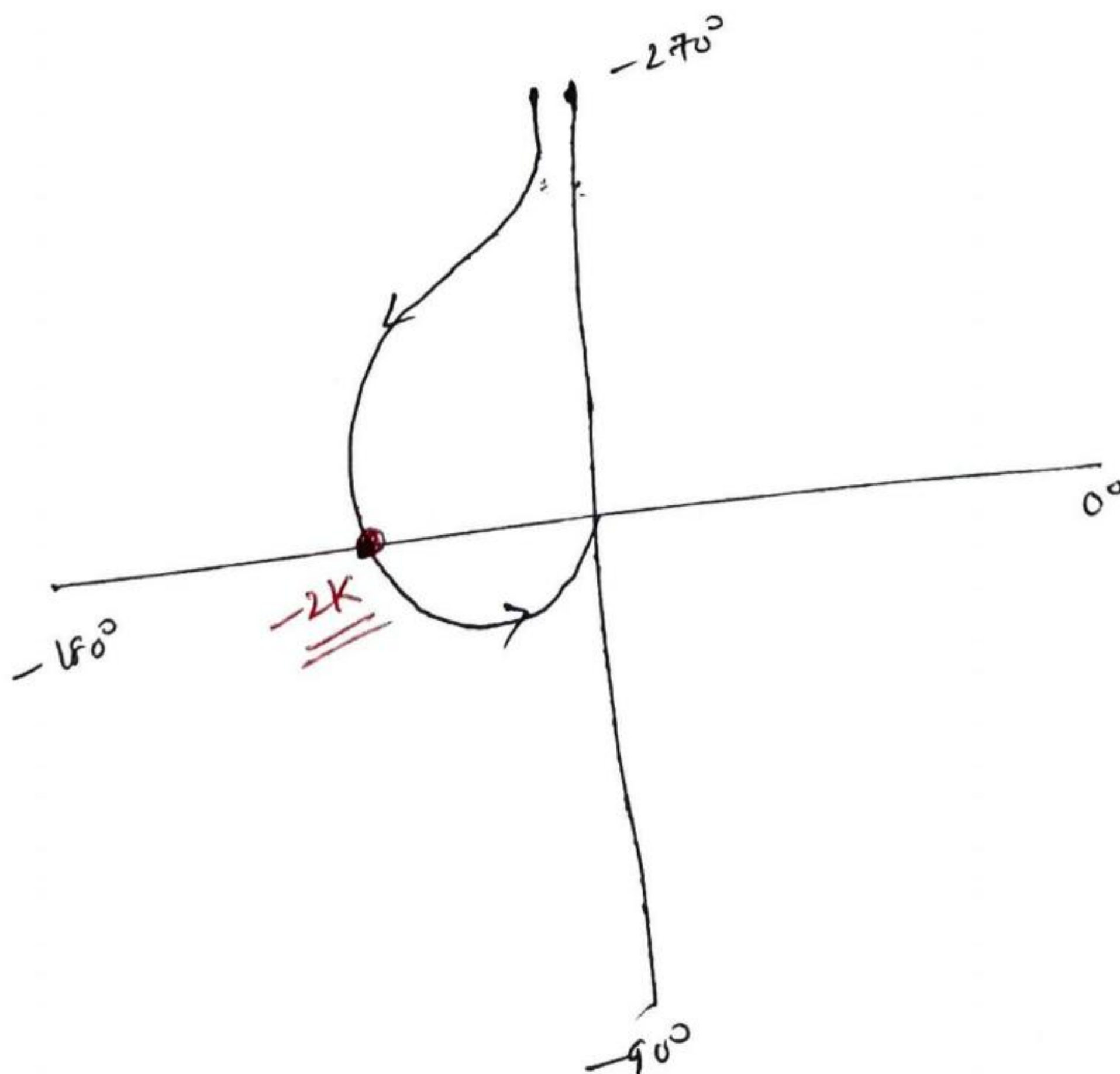
$$G(j\omega)H(j\omega) = \frac{K(1+j\omega)^2}{(j\omega)^3}$$

$$M = \frac{K(1+\omega^2)}{\omega^3}$$

$$\theta = -270^\circ + \tan^{-1}(\omega) + \tan^{-1}(\omega)$$

$$\begin{aligned} \omega=0, & \quad M_1 = \infty, \quad \theta_1 = -270^\circ \\ \omega=\infty, & \quad M_\infty = 0, \quad \theta_2 = -90^\circ \end{aligned}$$

E.D $\Rightarrow$ anti clock wise.
S.D $\Rightarrow$ anti clock wise.



$$G(s)H(s) = \frac{K(1+s)^2}{(s)^3}$$

$$= \cancel{2s} \frac{-2K}{s^2} + j \frac{K(1-s^2)}{s^3}$$

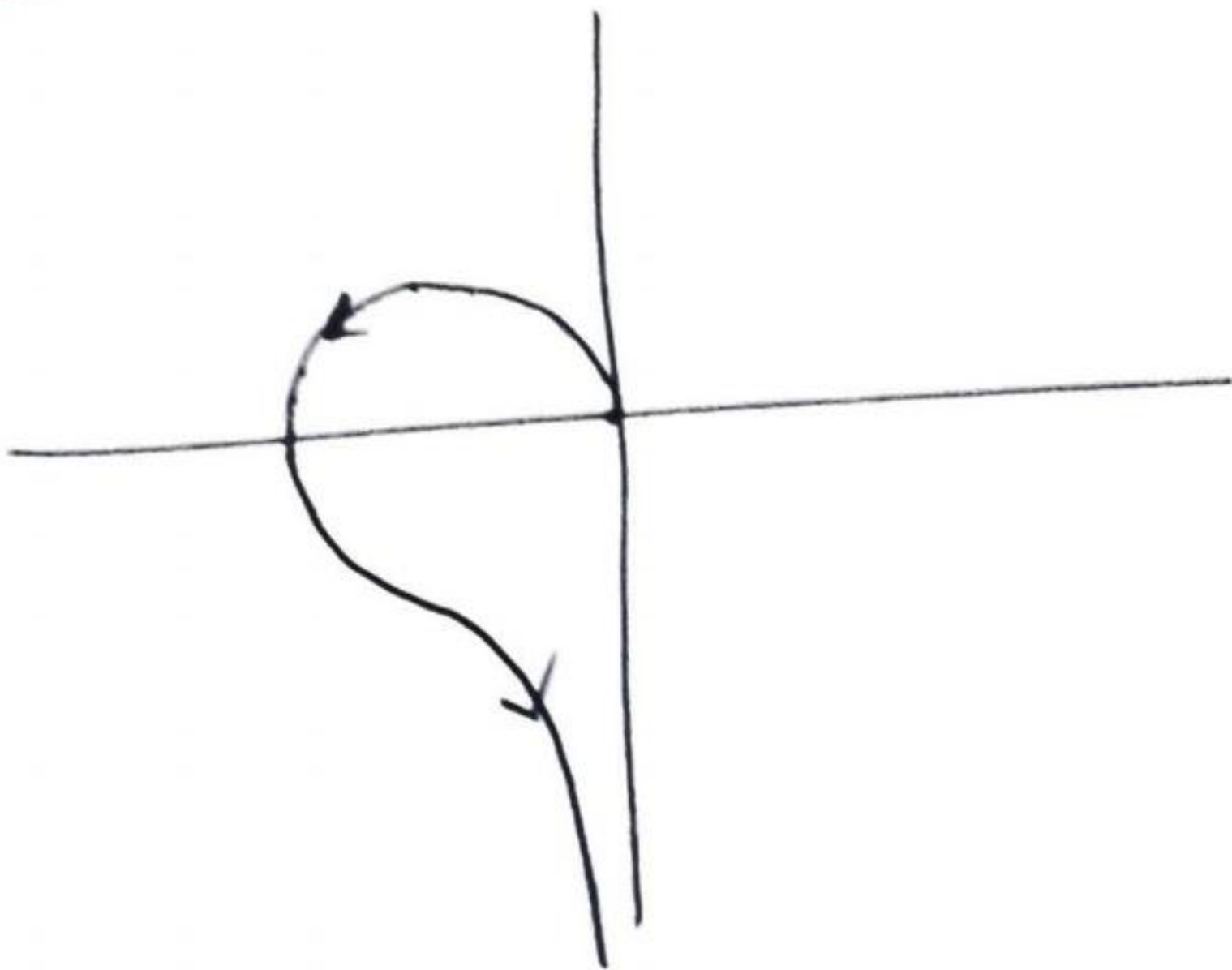
$$\text{Imag term} = 0$$

$$\frac{1-s^2}{s^3} = 0$$

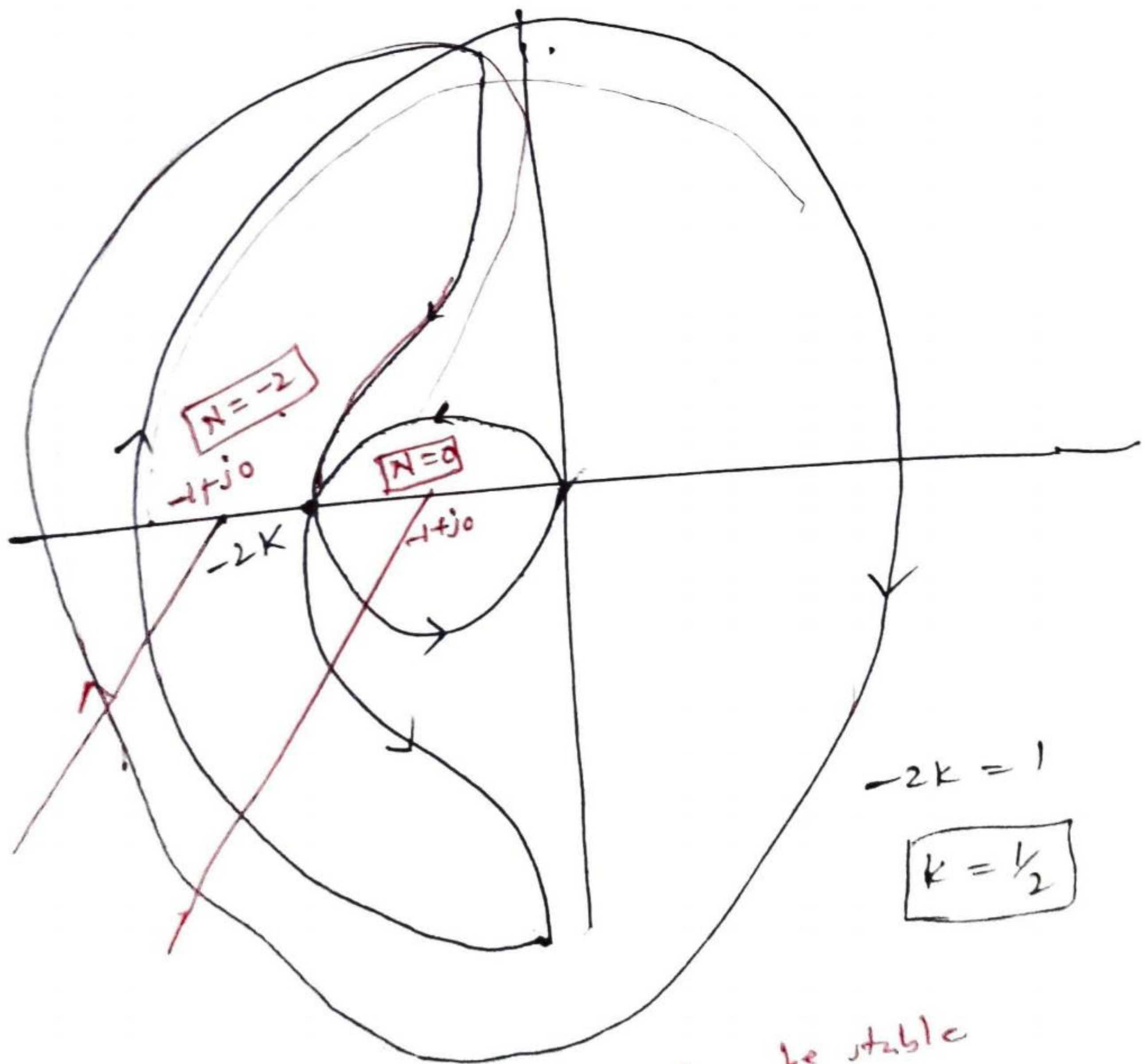
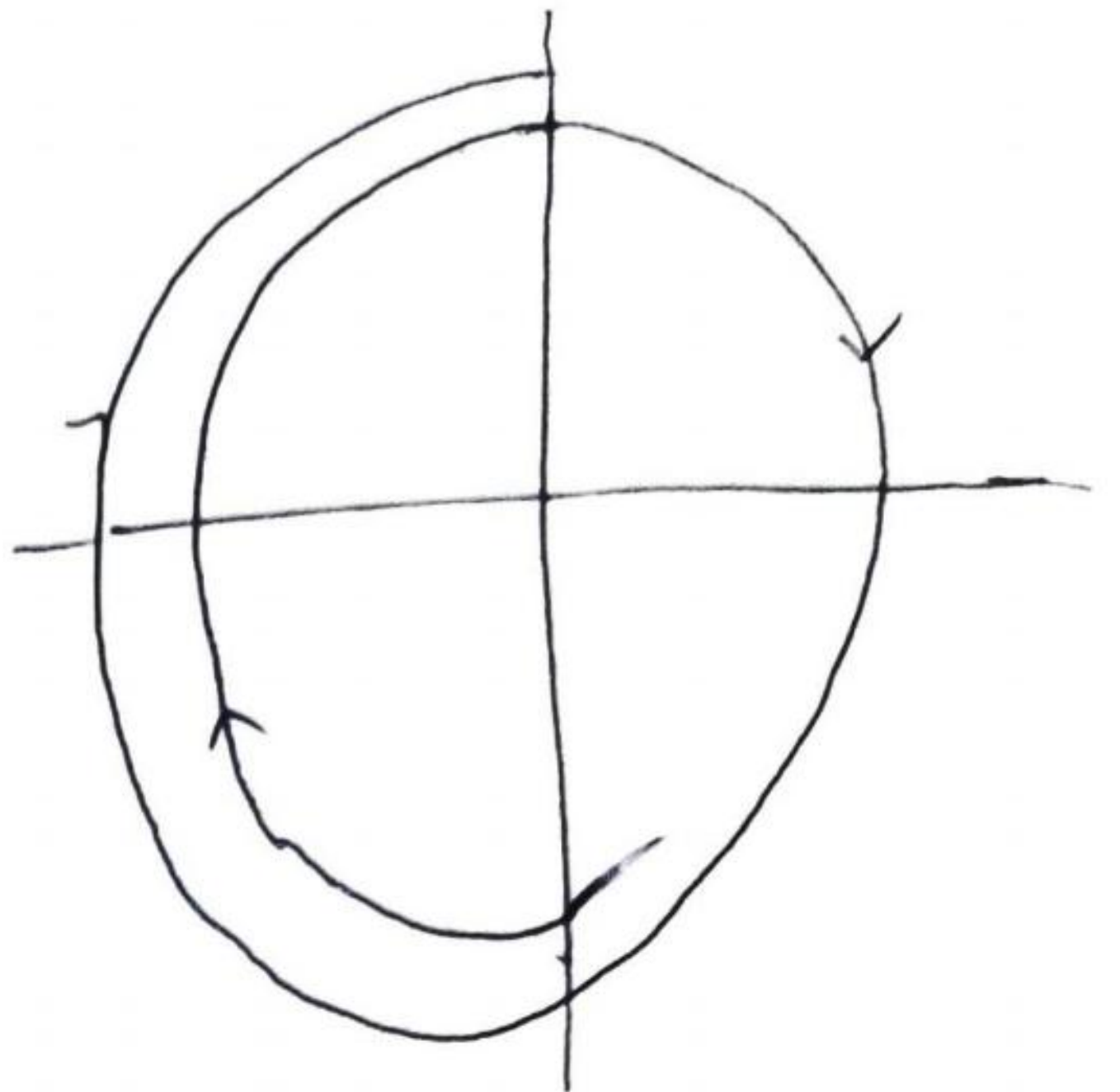
$$1 - \omega_{pc}^2 = 0 \Rightarrow \omega_{pc} = 1 \text{ rad/sec}$$

$$\begin{aligned} \text{At } \omega = \omega_{pc} \Rightarrow G(j\omega)H(j\omega) &= \frac{-2K}{\omega^2} + 0 \\ &= \underline{\underline{-2K}} \end{aligned}$$

Section C3



Section C4



$$-2K = 1$$

$$K = \frac{1}{2}$$

$$K < 0.5$$

$$K > 0.5$$

→ TO system to be stable

Classification of systems:

This classification is based on position of zeros but not poles. The poles always lie in left half of s-plane only.

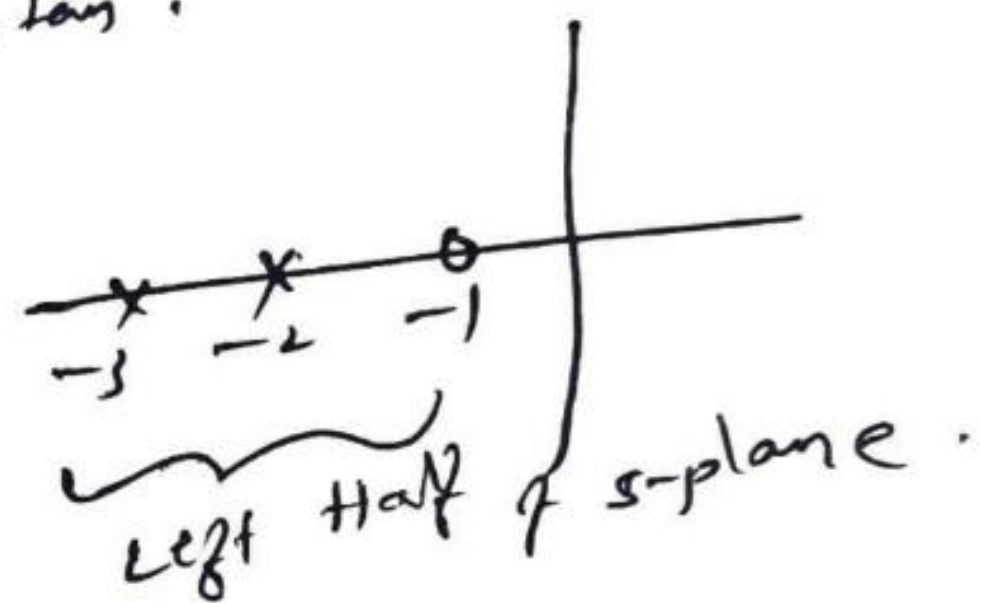
- i) Minimum-phase system.
- ii) All pass system.
- iii) Non minimum phase system.

1) Minimum phase system:

A system in which all the finite poles and finite zeros lie in the left half of s-plane. Then it is called minimum phase system.

Eg:

$$G(s) = \frac{s+1}{(s+2)(s+3)}$$

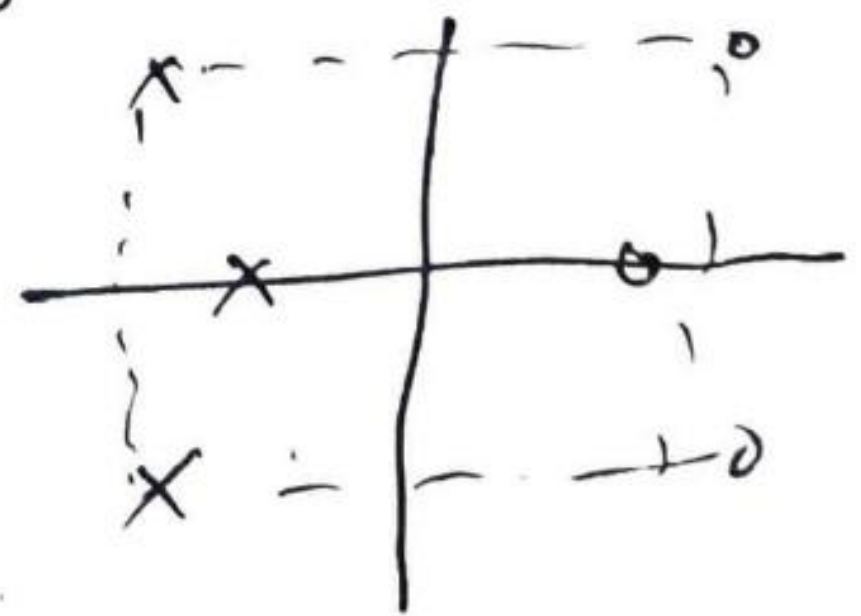


2) All pass system:

A system in which all the zeros lie on the right half of s-plane and all the poles lie on the left half of s-plane which are symmetrical about imaginary axis, then it is called All pass system. For all pass system the magnitude should always be '1'.

Eg:

$$G(s) = \frac{(s-1)(s^2+2s+2)}{(s+1)(s^2+2s+4)}$$

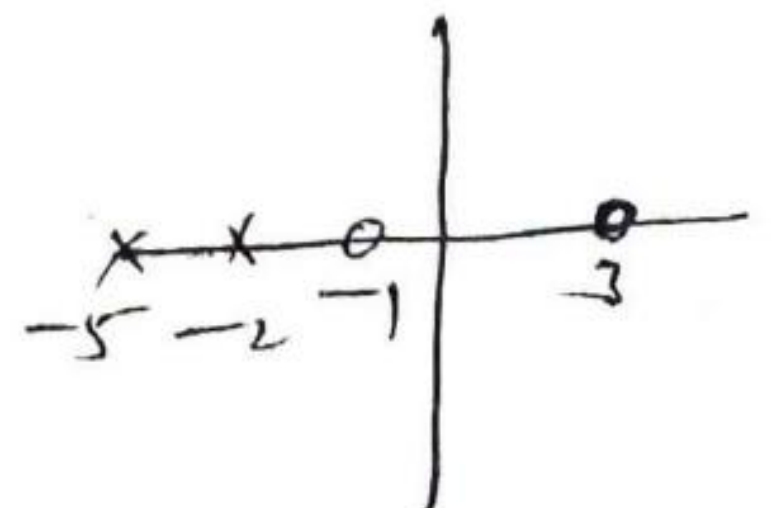


3) Non minimum phase system:

A system in which one or more zeros lie in the right half of s-plane and remaining all the poles and zeros lie in left half of s-plane. Then the system is called Non-min. phase system.

Eg:

$$G(s) = \frac{(s+1)(s-3)}{(s+2)(s+5)}$$



Compensation Techniques:

The compensator may be electrical, mechanical, hydraulic or other type of device or network. Usually the electrical network serves as compensator in many control systems. Different types of compensators are

- 1) Lag compensator
- 2) Lead compensator
- 3) Lag-lead compensator.

Use of compensator:

- i) when the system is absolutely unstable, then compensation is required to stabilize the system and also to meet desired performance.
- ii) when the system is stable, compensation is provided to obtain desired performance.

Lag compensator:

A compensator having the characteristics of a lag network is called lag compensator.

If a sinusoidal signal is applied to a lag network, then in steady state the o/p will have a phase lag w.r.t i/p.

Adv.

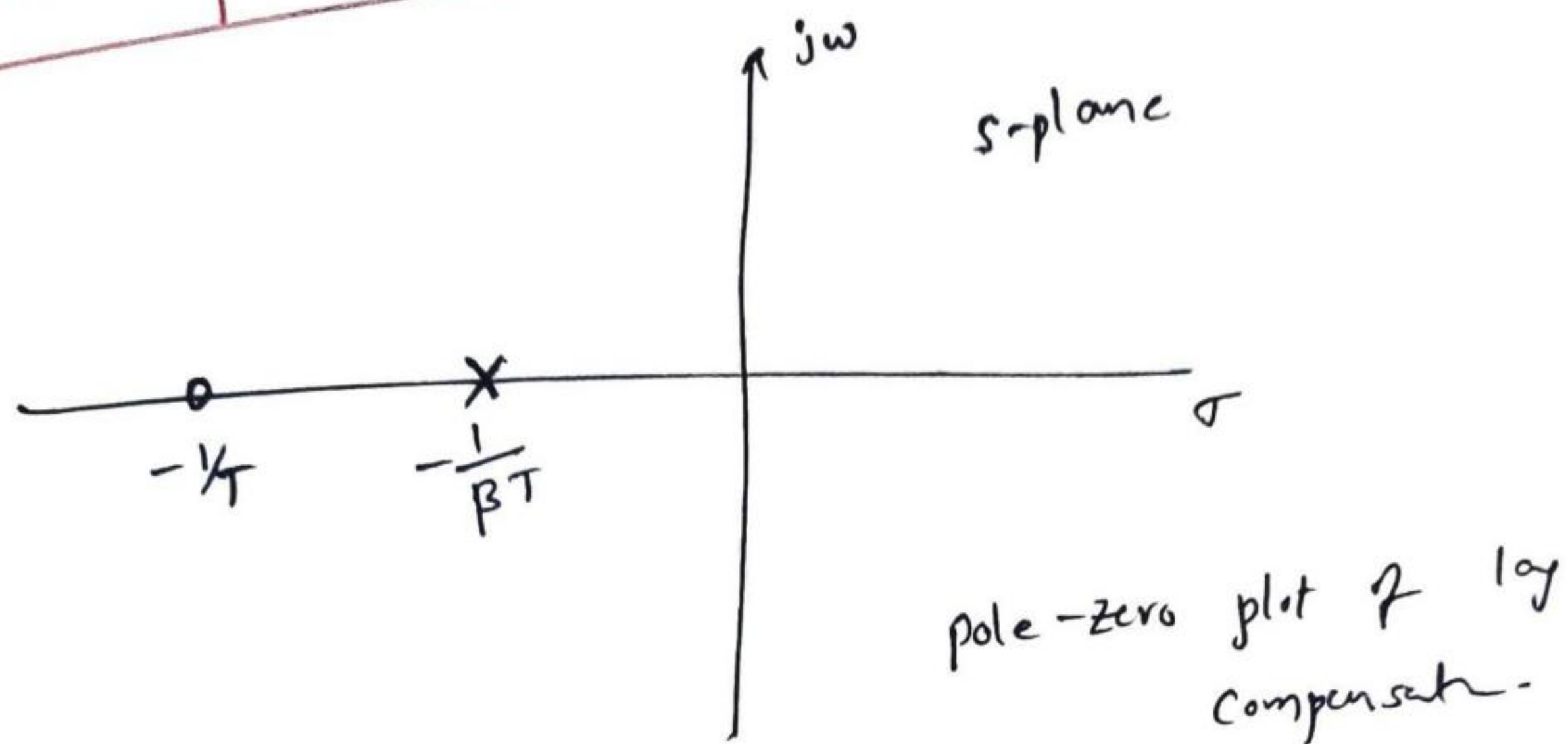
- i) It improves steady state performance of the system
- ii) It will shift ω_{gc} to a lower frequency point where phase margin is acceptable

Disadv

- i) It reduces bandwidth
- ii) slower response.

Lag compensator essentially a low pass filter and so high frequency noise signals are attenuated.

s-plane representation of lag compensator

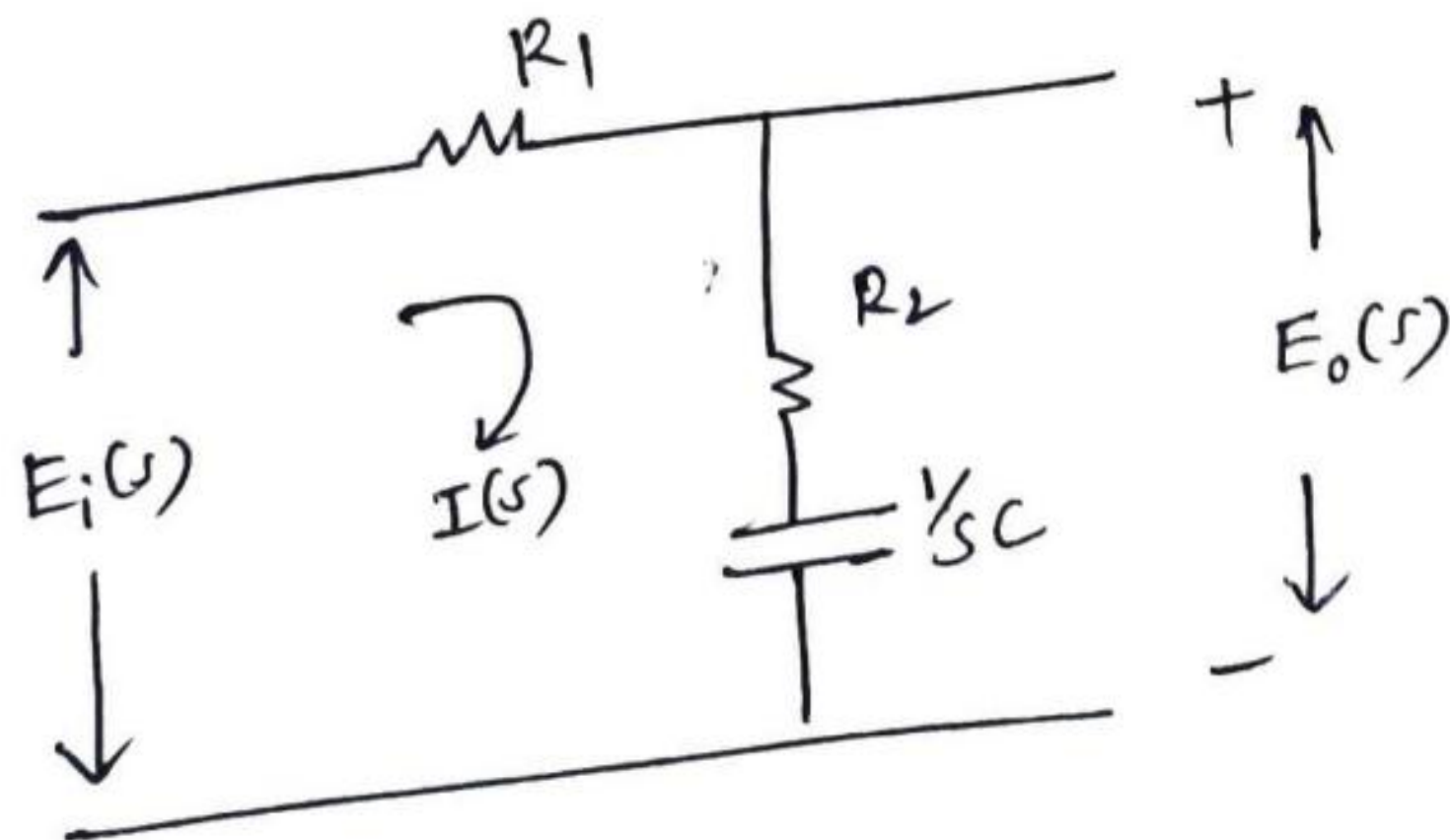


The general form of T.F of lag compensator is

$$G_c(s) = \frac{s+z_c}{s+p_c} = \frac{s+1/T}{s+1/\beta T}$$

Realization of lag compensator using electrical N/w

Lag compensator is realized by a simple R-C circuit



$$E_o(s) = \frac{E_i(s) [R_2 + 1/sC]}{R_1 + R_2 + 1/sC}$$

$$\frac{E_o(s)}{E_i(s)} = \frac{sCR_2 + 1}{sC(R_1 + R_2) + 1}$$

$$\frac{E_o(s)}{E_i(s)} = \frac{R_2 (s + 1/R_2 C)}{(R_1 + R_2) \left[s + \frac{1}{(R_1 + R_2) C} \right]}$$

$$= \frac{s + 1/R_2 C}{\left[\frac{R_1 + R_2}{R_2} \right] \left[s + \frac{1}{(R_1 + R_2) C} \right]}$$

$$= \frac{s + \frac{1}{R_2 C} \quad \underbrace{\hspace{1cm}}_T}{\underbrace{\frac{R_1 + R_2}{R_2}}_{\beta} \left[s + \underbrace{\frac{1}{(R_1 + R_2) C}}_{\frac{1}{\beta T}} \cdot \underbrace{\frac{1}{R_2 C}}_T \right]} \rightarrow (1)$$

But the T.F of lag compensation

$$G_c(s) = \frac{s + 1/T}{s + 1/\beta T} \rightarrow (2)$$

By comparing (1) & (2)

$$\frac{E_o(s)}{E_i(s)} = \frac{1}{\beta} \cdot \left[\frac{s + 1/T}{s + 1/\beta T} \right]$$

where ~~T = R_2 C~~ $T = R_2 C$, $\beta = R_1 + R_2 / R_2$

Frequency Response of lag compensator

consider the general form of lag compensator

$$G_c(s) = \left[\frac{s + 1/T}{s + 1/\beta T} \right] = \frac{(sT + 1) \cdot \beta T}{(s\beta T + 1)T}$$
$$= \frac{\beta(1 + sT)}{1 + s\beta T}$$

To get sinusoidal T.F. sub. $s = j\omega$

$$G_c(j\omega) = \frac{\beta(1 + j\omega T)}{(1 + j\omega\beta T)}$$

$$\text{When } \omega = 0, G_c(j\omega) = \beta$$

We can say that the lag compensator provides a dc gain of β ($\beta > 1$). If dc gain is not desirable then it can be eliminated by suitable compensation.

Let us assume that the gain β eliminated by a suitable attenuator N/O. The $G_c(j\omega)$ is given by

$$G_c(j\omega) = \frac{1 + j\omega T}{1 + j\omega\beta T}$$

$$|G_c(j\omega)| = \frac{\sqrt{1 + \omega^2 T^2}}{\sqrt{1 + \omega^2 \beta^2 T^2}}$$

$$\phi = \tan^{-1}(\omega T) - \tan^{-1}(\omega \beta T)$$

The sinusoidal TFR has two corner freq. ω_{c1}, ω_{c2}

where $\omega_{c1} = \frac{1}{\beta T}$, $\omega_{c2} = \frac{1}{T}$

Since $\beta T > T$, $\omega_{c1} < \omega_{c2}$

At very low freq. i.e. upto ω_{c1} , $\omega T \ll 1$
 and $\omega \beta T \ll 1$

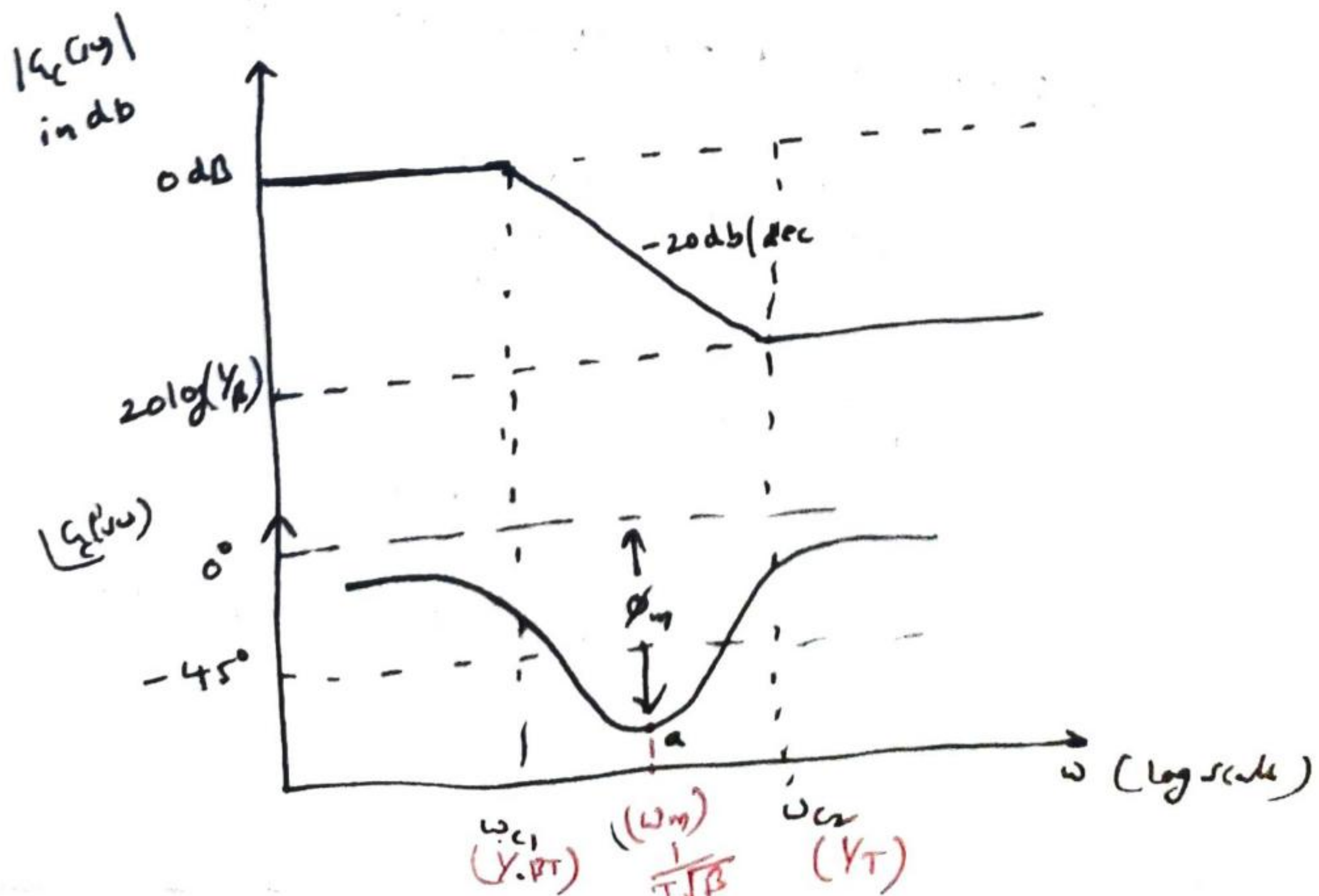
$\therefore A = 20 \log 1 = 0$

In the freq. range from ω_{c1} to ω_{c2} , $\omega T \ll 1$
 and $\omega \beta T \gg 1$

$$A = 20 \log \frac{1}{\sqrt{(\omega \beta T)^2}} = 20 \log \frac{1}{\omega \beta T} = -20 \log \omega \beta T$$

At very high freq, after ω_{c2} $\omega T \gg 1$, $\omega \beta T \gg 1$

$$A = 20 \log \frac{\sqrt{(\omega T)^2}}{\sqrt{(\omega \beta T)^2}} = 20 \log \frac{1}{\beta}$$



$$\text{Let } \phi = \angle G(j\omega)$$

$$\phi = \tan^{-1}(\omega T) - \tan^{-1}(\beta \omega T)$$

$$\text{As } \omega \rightarrow 0, \phi \rightarrow 0$$

$$\text{As } \omega \rightarrow \infty, \phi \rightarrow 0$$

$$\begin{aligned} \text{Frequency of max. phase lag } \omega_m &= \sqrt{\omega_{c1} \cdot \omega_{c2}} \\ &= \sqrt{\frac{1}{\beta T} \cdot \frac{1}{T}} \end{aligned}$$

$$\omega_m = \frac{1}{T\sqrt{\beta}}$$

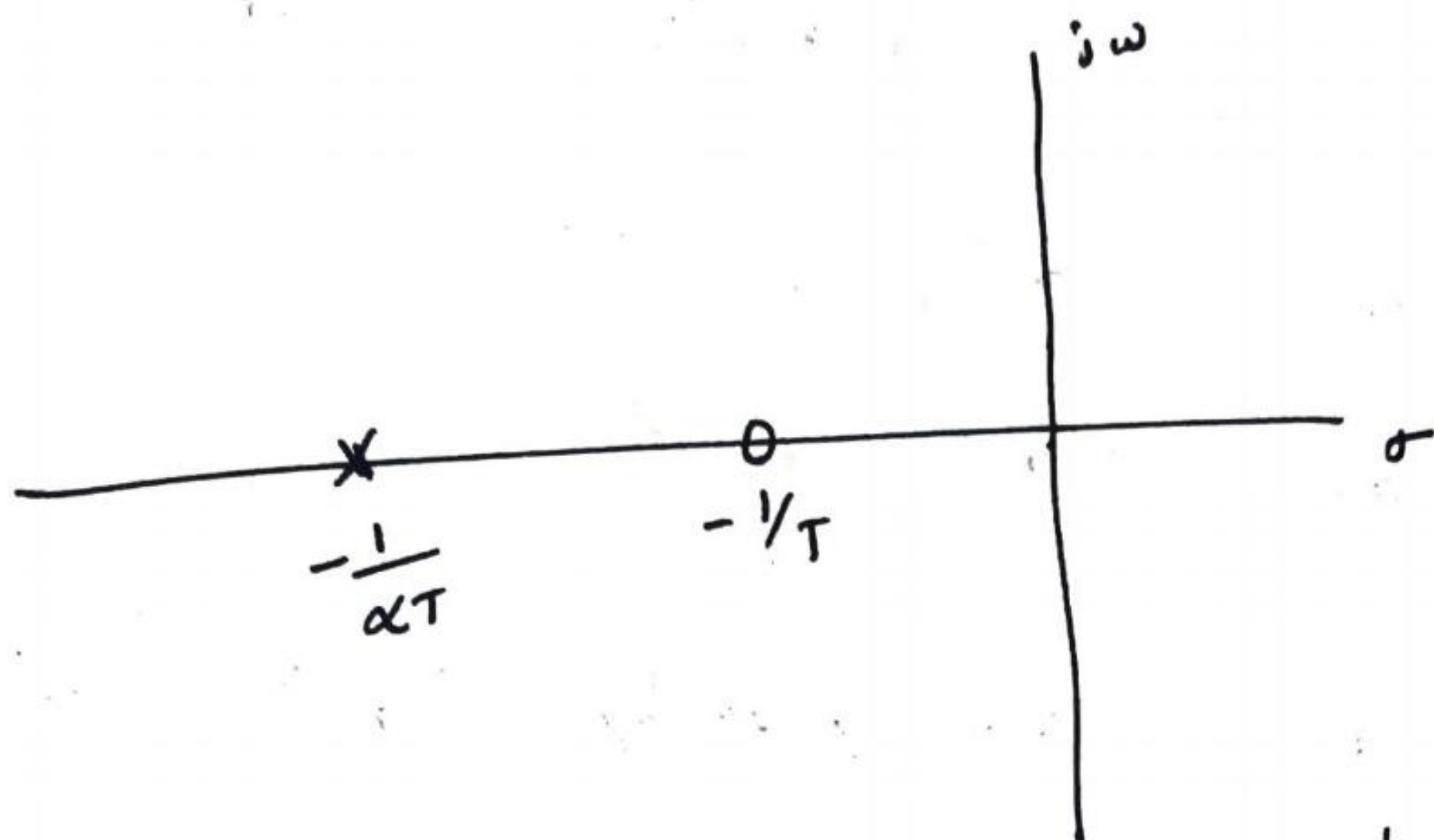
Lead Compensation

A compensator having the char. of a lead N/O is called lead compensator. If a sinusoidal signal is applied to the lead N/O, then in steady state the o/p will have a phase lead w.r.t. i/p.

The lead compensation increases Bandwidth which improves speed of the response and also reduces peak overshoot. It improves transient response.

Lead compensator is basically a high pass filter and it amplifies high freq. noise signals.

s-plane Representation of lead compensator



Pole-zero plot of lead compensator

The lead compensator has a zero at $s = -1/T$ and a pole at $s = -1/\alpha T$. Here $\alpha < 1$, so the zero is closer to the origin than the pole. The general form of T.F of lead compensator is given by

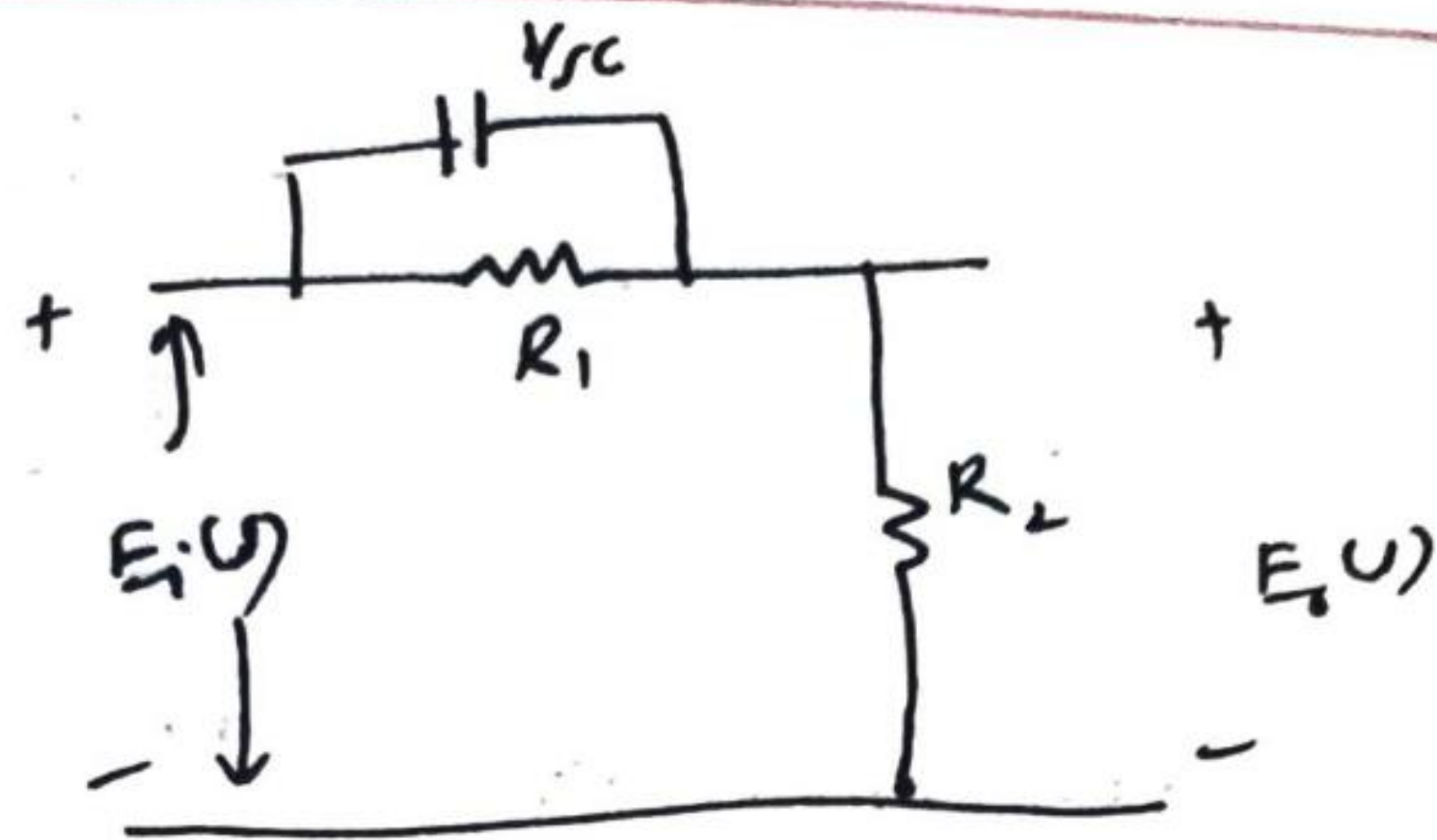
$$G_c(s) = \frac{s + z_c}{s + p_c} = \left[\frac{s + 1/T}{s + 1/\alpha T} \right]$$

where $T > 0$, $\alpha < 1$

zero is $z_c = 1/T$

pole is $p_c = 1/\alpha T$

Realization of lead compensator using Electrical N/w



By voltage division rule

$$\text{output } V_o \quad E_o(s) = \frac{E_i(s) \cdot R_2}{R_2 + [R_1 \parallel 1/sC]}$$

$$E_o(s) = E_i(s) \cdot \frac{R_2}{R_2 + \frac{R_1 \cdot 1/sC}{R_1 + 1/sC}}$$

$$= E_i(s) \cdot \frac{R_2}{\frac{R_2(R_1 sC + 1) + R_1}{R_1 sC + 1}}$$

$$\text{T.F.} \quad \frac{E_o(s)}{E_i(s)} = \frac{R_2(R_1 sC + 1)}{[R_1 R_2 sC + R_2 + R_1]}$$

$$= \frac{R_1 C / R_2 \left[s + \frac{1}{R_1 C} \right]}{R_1 R_2 C \left[s + \frac{(R_1 + R_2)}{R_1 C R_2} \right]}$$

$$\frac{E_o(s)}{E_i(s)} = \frac{s + 1/R_1 C}{s + \left[\frac{1}{R_2/R_1 + R_2} \right] \cdot \frac{1}{R_1 C}} \rightarrow (1)$$

General form of lead compensator is

$$G_c(s) = \frac{s + 1/T}{s + 1/\alpha T} \rightarrow (2)$$

By comparing ① & ② we get

$$\frac{E_o(s)}{E_i(s)} = \frac{s + \frac{1}{T}}{s + \frac{1}{\alpha T}}$$

where $T = R_1 C$ and $\alpha = \frac{R_2}{R_1 + R_2}$

Frequency response of lead compensator

By comparing ① + ② we get

$$\frac{E_o(s)}{E_i(s)} = \frac{s + 1/T}{s + \frac{1}{\alpha T}}$$

where $T = R_1 C$ and $\alpha = \frac{R_2}{R_1 + R_2}$

Frequency response of lead compensator

consider the general form of lead compensator

$$G_c(s) = \frac{s + 1/T}{s + 1/\alpha T} = \frac{(1 + sT)/T}{(s\alpha T + 1)/\alpha T}$$

$$= \frac{\alpha (1 + sT)}{(1 + s\alpha T)}$$

To convert it into sinusoidal T-F put $s = j\omega$

$$G_c(j\omega) = \frac{\alpha (1 + j\omega T)}{(1 + j\omega \alpha T)}$$

when $\omega = 0$, $G_c(j\omega) = \alpha$

From above equation we can say that the lead compensator provides an attenuation of ' α ' ($\alpha < 1$). If attenuation is not desirable then it can be eliminated by a suitable ampl^r.

then

$$G_c(j\omega) = \frac{(1 + j\omega T)}{(1 + j\omega \alpha T)}$$

$$\text{Magnitude } |G_c(j\omega)| = \frac{\sqrt{1+(\omega T)^2}}{\sqrt{1+(\alpha\omega T)^2}}$$

$$\text{phase } \phi = \tan^{-1}(\omega T) - \tan^{-1}(\alpha\omega T)$$

Sinusoidal T.F by two corner frequencies ω_{c1} and ω_{c2}

$$\text{Here } \omega_{c1} = \frac{1}{T} \text{ and } \omega_{c2} = \frac{1}{\alpha T}$$

$$\text{since } T > \alpha T, \quad \omega_{c1} < \omega_{c2}$$

$$\text{Let } A = |G(\omega)| \text{ in db} = 20 \log \frac{\sqrt{1+(\omega T)^2}}{\sqrt{1+(\alpha\omega T)^2}}$$

At very low freq. i.e. upto ω_{c1}
 $\omega T \ll 1$ and $\omega \alpha T \ll 1$

$$A = 20 \log 1 = 0 \text{ db}$$

In the frequency range from ω_{c1} to ω_{c2} ,
 $\omega T \gg 1$ and $\omega \alpha T \ll 1$

$$A \approx 20 \log \sqrt{(\omega T)^2} = 20 \log \omega T$$

At high freq. i.e. after ω_{c2} , $\omega T \gg 1$ and $\omega \alpha T \gg 1$

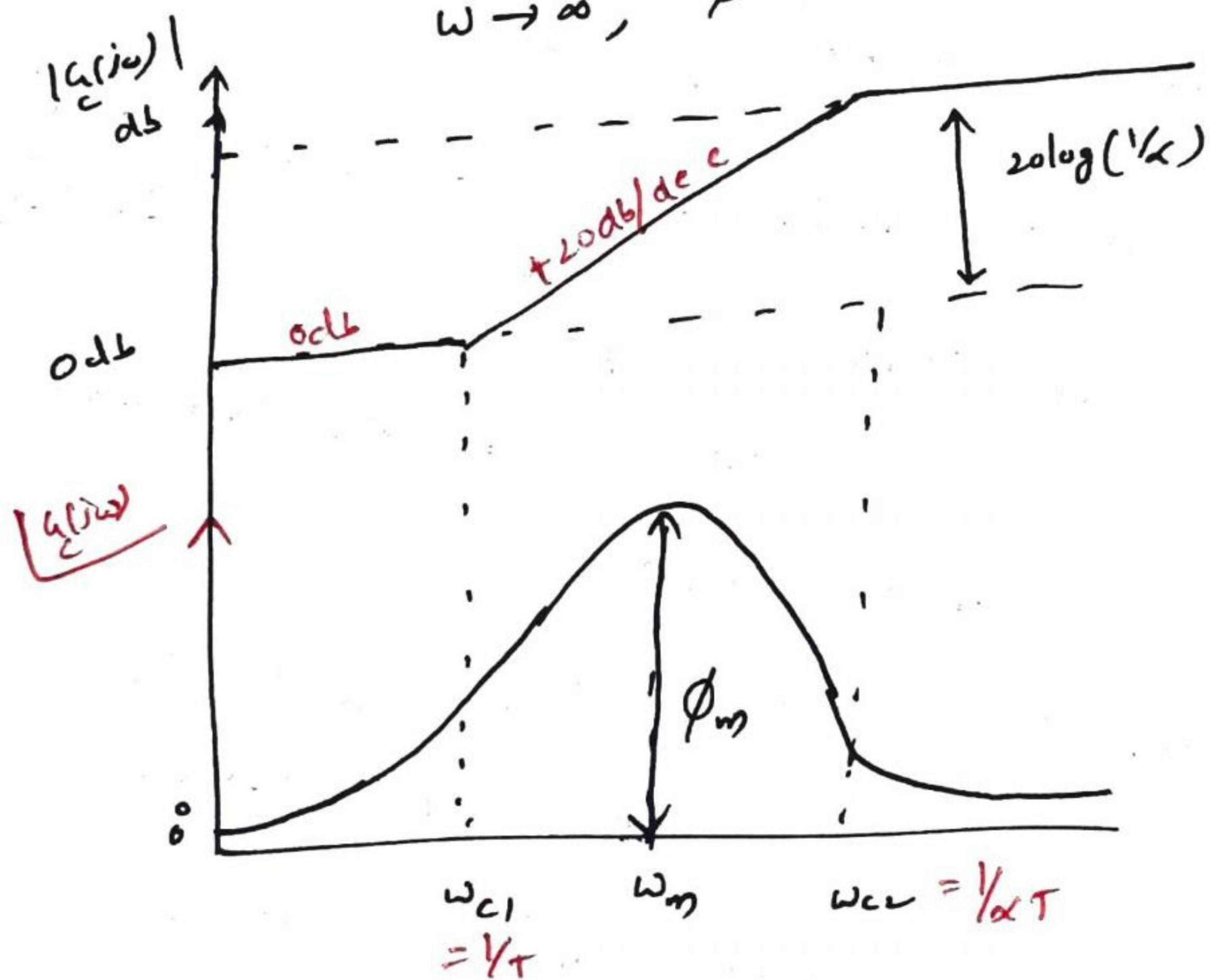
$$A \approx 20 \log \frac{\sqrt{(\omega T)^2}}{\sqrt{(\omega \alpha T)^2}} = 20 \log \frac{1}{\alpha}$$

The magnitude plot of $G_c(\omega)$ is a straight line through 0dB upto ω_{c1} , then it has slope of +20dB/dec upto ω_{c2} and after ω_{c2} it is a straight line with constant gain of $20\log(1/\alpha)$

Let $\phi = \angle G_c(j\omega) = \tan^{-1}(\omega T) - \tan^{-1}(\alpha \omega T)$

As $\omega \rightarrow 0$, $\phi \rightarrow 0$

$\omega \rightarrow \infty$, $\phi \rightarrow 0$



Freq of max. phase lead $\omega_m = \sqrt{\omega_{c1} \cdot \omega_{c2}}$

$$= \sqrt{\frac{1}{T} \cdot \frac{1}{\alpha T}}$$

$$\boxed{\omega_m = \frac{1}{T\sqrt{\alpha}}}$$

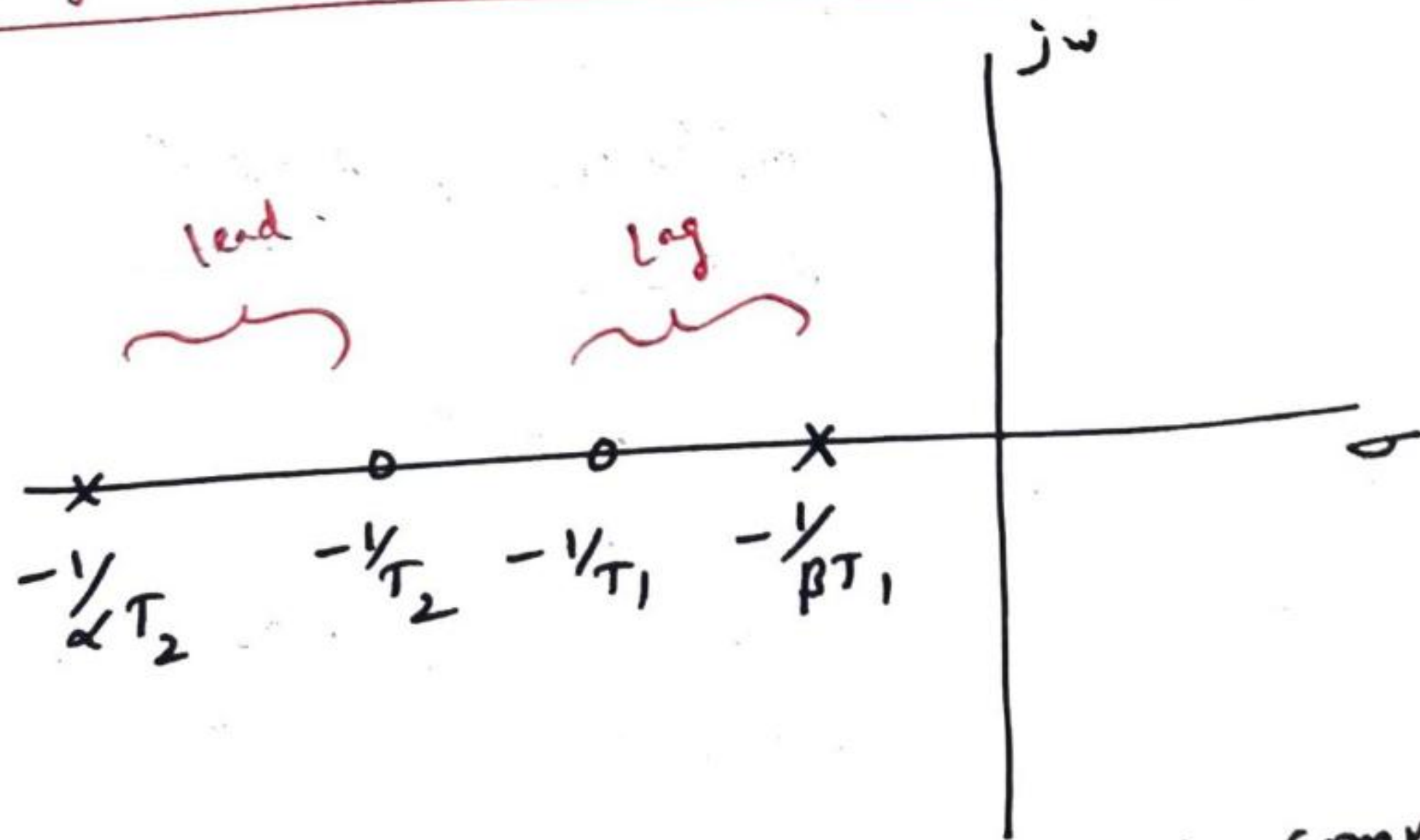
$$\phi_m = \tan^{-1}\left[\frac{1-\alpha}{2\sqrt{\alpha}}\right] \quad \text{where } \alpha = \frac{1 - \sin \phi_m}{1 + \sin \phi_m}$$

Lag-lead compensator?

A compensator having the characteristics of lag-lead N/w is called lag-lead compensator. In lag-lead N/w when sinusoidal signal is applied, both phase lag and phase lead occurs in the o/p, but in different freq. regions.

A lag lead compensator combines the advantages of lag and lead compensations. lag-lead compensator possess two poles and two zeros and compensation increases the order of the system by two.

s-plane Representation of lag-lead compensator:



pole-zero plot of lag-lead compensator.

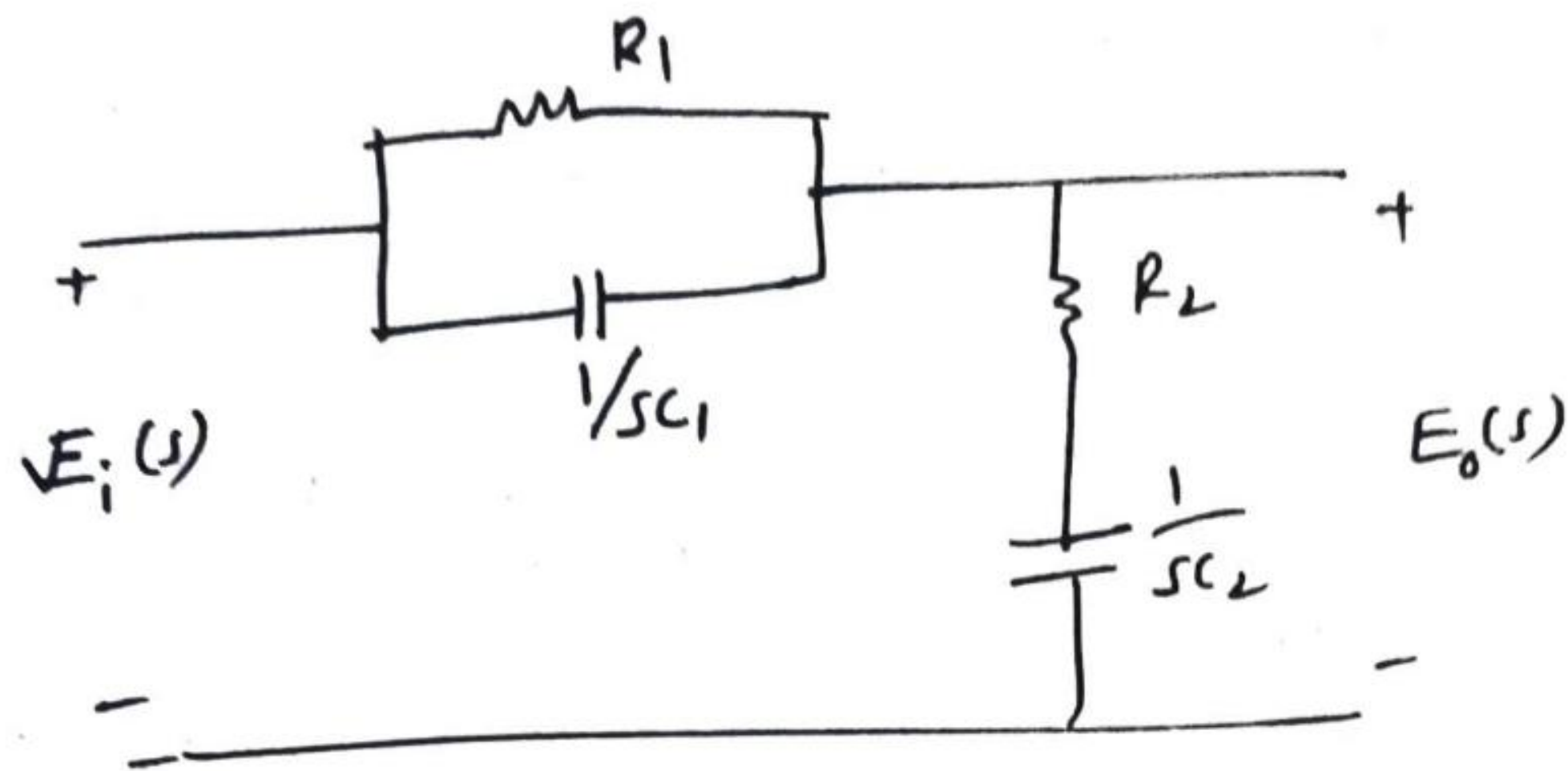
T.F of lag-lead compensator is given by

$$G_c(s) = \underbrace{\left[\frac{s + 1/T_1}{s + 1/\beta T_1} \right]}_{\text{lag section}} \underbrace{\left[\frac{s + 1/T_2}{s + 1/\alpha T_2} \right]}_{\text{lead section}}$$

where $\beta > 1$ and $0 < \alpha < 1$

Realization of lag-lead compensator using electrical

Nw :



By voltage division Rule.

$$E_o(s) = \frac{E_i(s) \cdot \left[R_2 + \frac{1}{sC_2} \right]}{\left[R_1 \parallel \frac{1}{sC_1} \right] + \left[R_2 + \frac{1}{sC_2} \right]}$$

$$= \frac{E_i(s) \left[R_2 + \frac{1}{sC_2} \right]}{\frac{R_1 \cdot \frac{1}{sC_1}}{R_1 + \frac{1}{sC_1}} + \frac{sR_2C_2 + 1}{sC_2}}$$

$$\frac{E_o(s)}{E_i(s)} = \frac{\frac{sR_2C_2 + 1}{sC_2}}{\frac{R_1}{sR_1C_1 + 1} + \frac{sR_2C_2 + 1}{sC_2}}$$

$$= \frac{(sR_2C_2 + 1)(sR_1C_1 + 1)}{sR_1C_2 + (sR_1C_1 + 1)(sR_2C_2 + 1)}$$

$$= \frac{R_1C_1R_2C_2 \left[s + \frac{1}{R_1C_1} \right] \left[s + \frac{1}{R_2C_2} \right]}{sR_1C_2 + R_1C_1R_2C_2 \left[s + \frac{1}{R_1C_1} \right] \left[s + \frac{1}{R_2C_2} \right]}$$

$$\frac{E_o(s)}{E_i(s)} = \frac{(s + 1/R_1 C_1)(s + 1/R_2 C_2)}{\frac{s}{R_2 C_1} + \left[s + \frac{1}{R_1 C_1}\right]\left[s + \frac{1}{R_2 C_2}\right]}$$

$$= \frac{\left[s + \frac{1}{R_1 C_1}\right]\left[s + \frac{1}{R_2 C_2}\right]}{s^2 + s\left[\frac{1}{R_1 C_1} + \frac{1}{R_2 C_2} + \frac{1}{R_2 C_1}\right] + \frac{1}{R_1 R_2 C_1 C_2}}$$

But T.F of lag-lead compensator is given by

$$G_c(s) = \frac{(s + 1/T_1)(s + 1/T_2)}{(s + 1/\alpha T_1)(s + 1/\alpha T_2)} \rightarrow (1)$$

$$G_c(s) = \frac{\left[s + \frac{1}{R_1 C_1}\right]\left[s + \frac{1}{R_2 C_2}\right]}{s^2 + s\left[\frac{1}{R_1 C_1} + \frac{1}{R_2 C_2} + \frac{1}{R_2 C_1}\right] + \frac{1}{R_1 R_2 C_1 C_2}} \rightarrow (2)$$

By comparing (1) & (2)

$$T_1 = R_1 C_1$$

$$T_2 = R_2 C_2$$

$$R_1 R_2 C_1 C_2 = \alpha T_1 T_2$$

$$\frac{1}{R_1 C_1} + \frac{1}{R_2 C_2} + \frac{1}{R_2 C_1} = \frac{1}{\alpha T_1} + \frac{1}{\alpha T_2}$$

$$\alpha \beta = \frac{R_1 R_2 C_1 C_2}{T_1 T_2}$$

Frequency Response of lag-lead compensator.

$$G_c(s) = \frac{(s + 1/T_1)(s + 1/T_2)}{(s + 1/\beta T_1)(s + 1/\alpha T_2)} = \alpha \beta \frac{(1 + sT_1)(1 + sT_2)}{(1 + s\beta T_1)(1 + s\alpha T_2)}$$

$$G_c(j\omega) = \alpha \beta \frac{(1 + j\omega T_1)(1 + j\omega T_2)}{(1 + j\omega\beta T_1)(1 + j\omega\alpha T_2)}$$

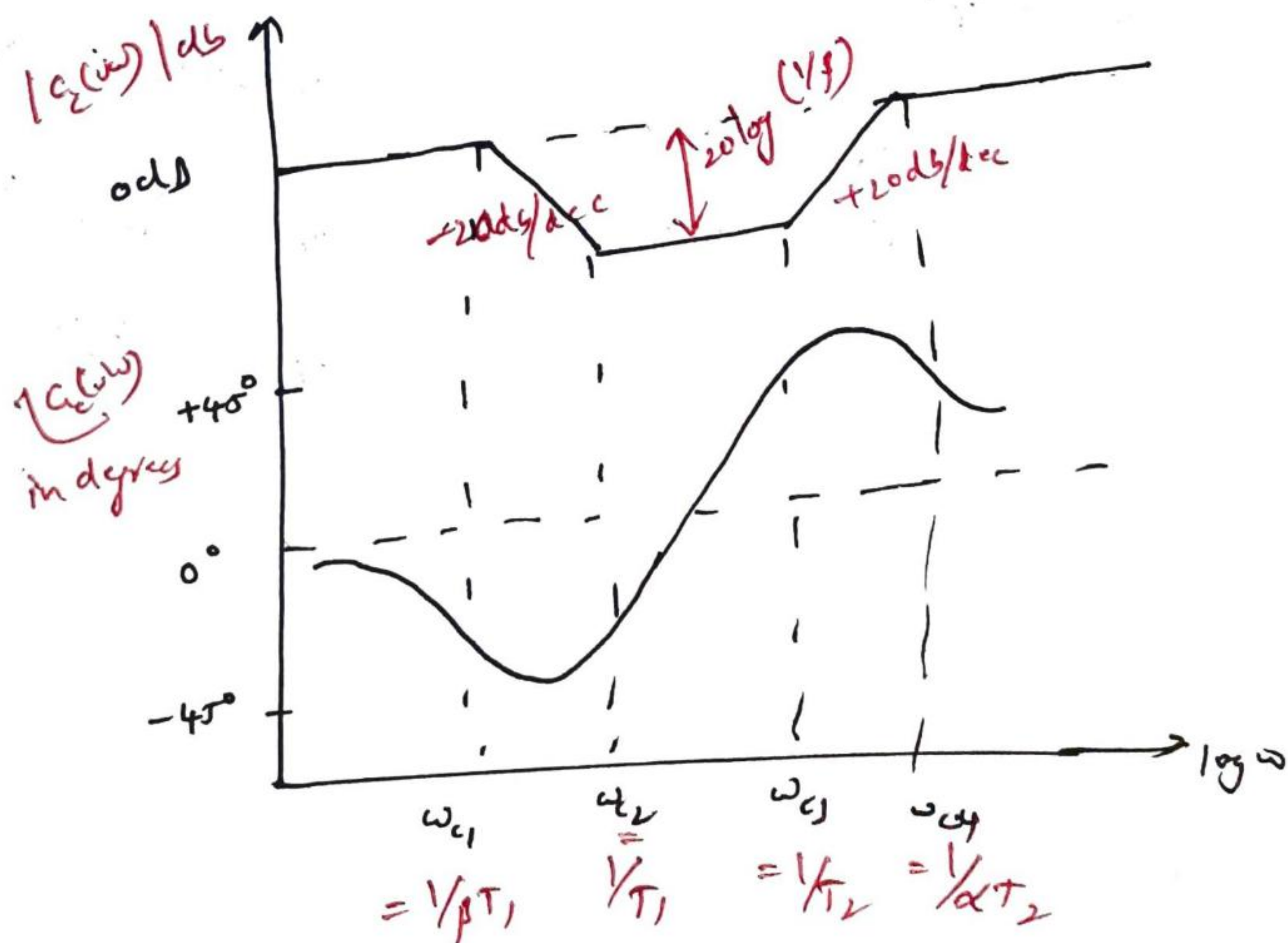
If $\alpha\beta = 1$, $G_c(j\omega) = \frac{(1 + j\omega T_1)(1 + j\omega T_2)}{(1 + j\omega\beta T_1)(1 + j\omega\alpha T_2)}$

The singuilar T.F has 4 corner freq. $\omega_{c1}, \omega_{c2}, \omega_{c3}, \omega_{c4}$

where $\omega_{c1} < \omega_{c2} < \omega_{c3} < \omega_{c4}$

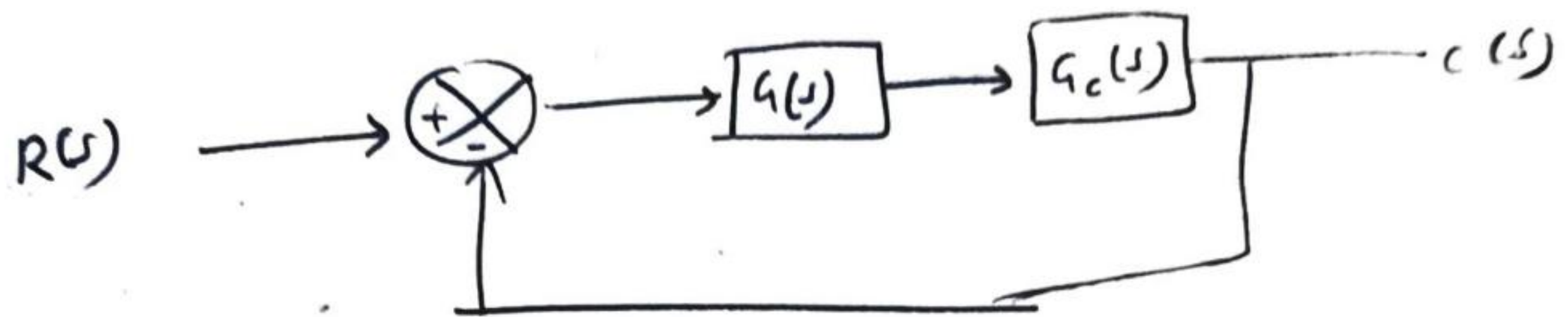
Here $\omega_{c1} = \frac{1}{\beta T_1}$, $\omega_{c2} = \frac{1}{T_1}$, $\omega_{c3} = \frac{1}{T_2}$, $\omega_{c4} = \frac{1}{\alpha T_2}$

By an analysis of (similar) of lag and lead compensator, we can draw bode plot of lag-lead compensator



PID Controllers:

A controller with T.F $G_c(s)$ can be introduced in cascade with OLTF, $G(s)$, to modify the transient and steady state response of the system.



The different types of controllers are

- 1) proportional controller
- 2) P-I controller
- 3) P-D controller
- 4) P-I-D controller.

The proportional controller improves steady state tracking accuracy and relative stability.

The P-I controller reduces steady state error and it increases the order and type number by one.

The PD controller increases damping of the system which results in reducing peak overshoot.

The PID Controller is a device which produces an output signal, $u(t)$ consisting of three terms - one is proportional to i/p signal $e(t)$, another one proportional to integral of input signal and the third one is proportional to derivative of the input signal $e(t)$.

$$\text{In PID controller } u(t) \propto \left[e(t) + \int e(t) dt + \frac{d}{dt} e(t) \right]$$

$$\therefore u(t) = K_p e(t) + K_I \int e(t) dt + K_D \frac{d}{dt} e(t)$$

on taking L.T we get,

$$\begin{aligned} U(s) &= K_p \cdot E(s) + K_I \frac{E(s)}{s} + K_D \cdot s \cdot E(s) \\ &= E(s) \left[K_p + \frac{K_I}{s} + K_D \cdot s \right] \end{aligned}$$

$$\text{T.F of PID controller is } G_c(s) = K_p + \frac{K_I}{s} + K_D \cdot s.$$

The PID Controller has the combined effect of all three control actions - i.e. proportional, integral and derivative control action. Hence, PID controller stabilizes the gain, reduces the steady state error and peak overshoot of the system. i.e. it will improve both transient and steady state response of the system.

Unit - 5

State space Analysis of continuous systems.

The T-F approach of representing a system is attractive because it provides with simple and powerful analysis and design techniques. But it has certain drawbacks. They are.

- 1) The T-F describes only the zero state response of a system. Hence ^{when} we ~~can~~ use the T-F, we assumed that the system is initially relaxed.
- 2) It describes only relation b/w input and o/p, but does not provide any information about the internal state of the system.
- 3) It is limited to single i/p and single o/p (SISO) systems.
- 4) It is applicable only for LTI systems.

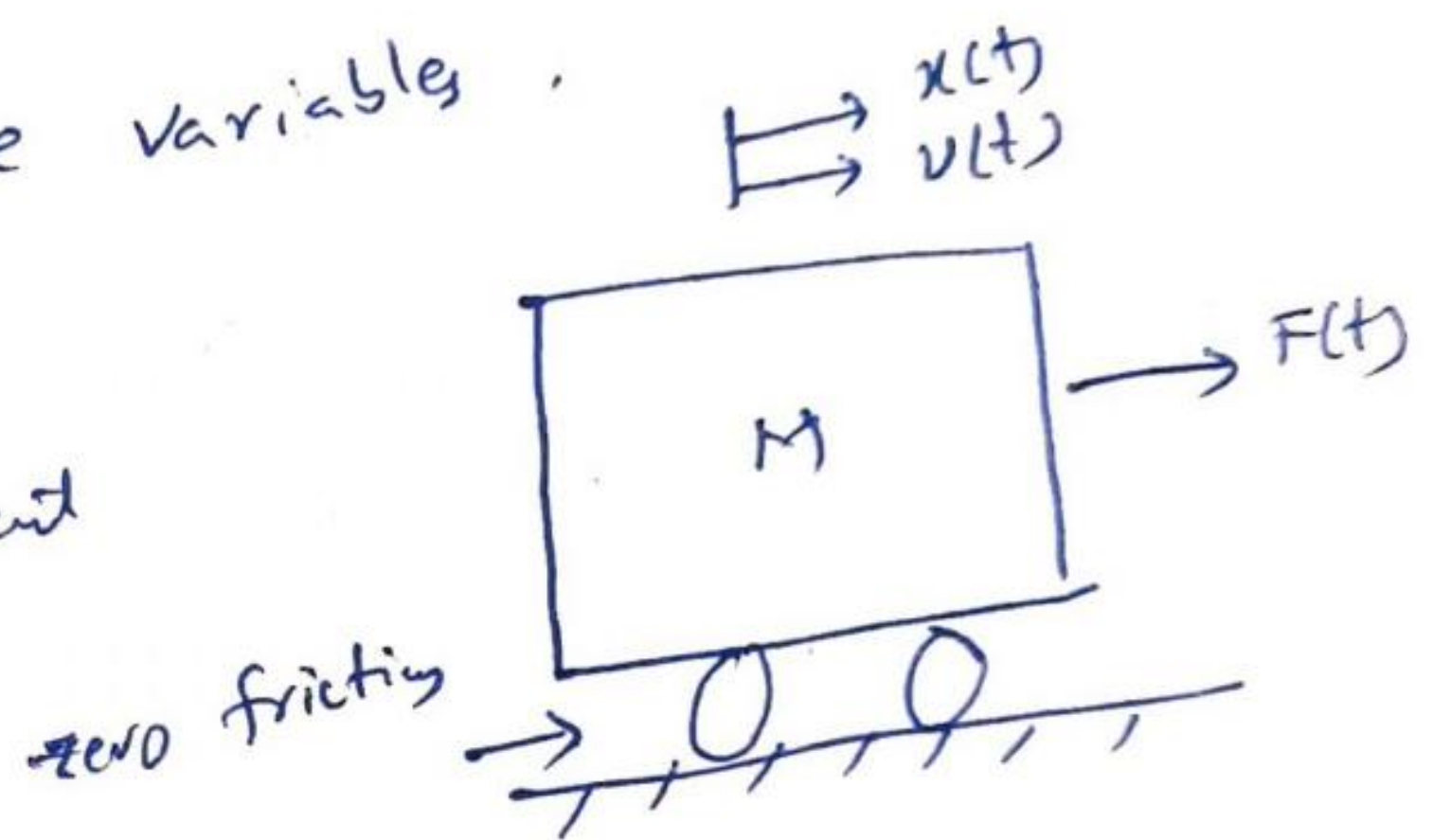
The above drawbacks in transfer function modelling can be overcome by an other method, known as state variable approach. It is a time domain approach can be applied for analysis and design of linear or non linear, time invariant or variant, Multi input multi-output (MIMO) systems. This can be applied even if the initial state of the system is non zero.

It describes not only i/p and o/p relationship, but also the internal state of the system.

concepts of state, state variable and state model:

Consider a mechanical system which is having mass (m) and force $f(t)$ acting upon mass element then the system utilizes three types of variable called input, output and state variable.

Force acting on mass element is given by



$$F(t) = M \cdot a$$

$$= M \cdot \frac{dv(t)}{dt}$$

$$v(t) = \int_{-\infty}^t \frac{F(t)}{M} \cdot dt$$

$$v(t) = \frac{1}{M} \int_{-\infty}^t F(t) \cdot dt$$

$$v(t) = \frac{1}{M} \left[\int_{-\infty}^{t_0} F(t) \cdot dt + \int_{t_0}^t F(t) \cdot dt \right]$$

$$= v(t_0) + \frac{1}{M} \int_{t_0}^t F(t) \cdot dt \quad \text{--- (1)}$$

But the output displacement $x(t)$ is given by

$$x(t) = \int_{-\infty}^t v(t) \cdot dt = \int_{-\infty}^{t_0} v(t) \cdot dt + \int_{t_0}^t v(t) \cdot dt$$

$$x(t) = x(t_0) + \int_{t_0}^t \left[v(t_0) + \frac{1}{M} \int_{t_0}^t F(t) \cdot dt \right]$$

[∵ From (1)]

$v(t) = \frac{dx(t)}{dt}$

$$x(t) = x(t_0) + v(t_0)[t=t_0] + \frac{1}{m} \int_{t_0}^t f(t) \cdot dt$$

$\downarrow$ $\downarrow$ $\downarrow$
 o/p initial o/p input

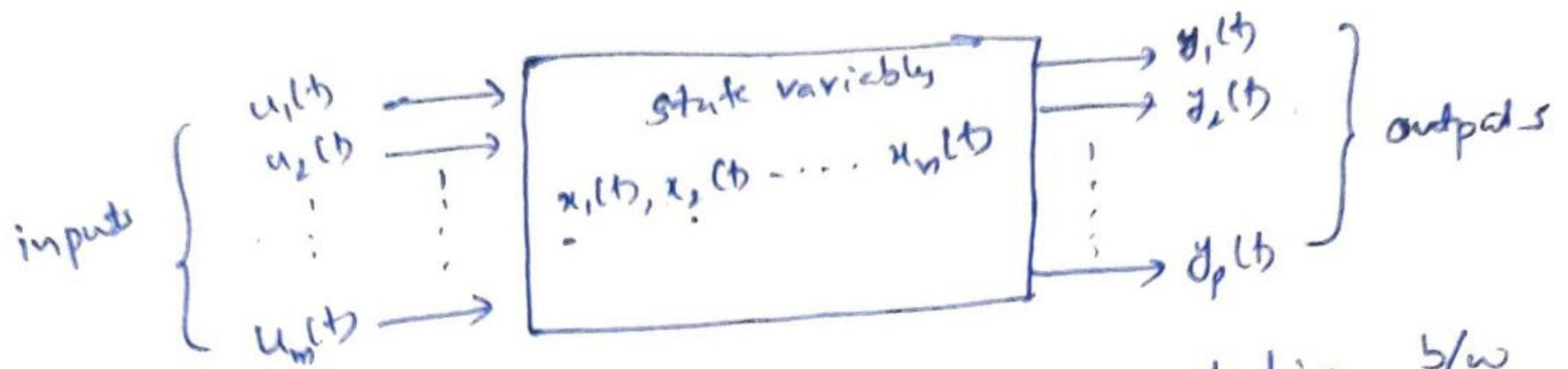
From above equation, we can observe that the displacement $x(t)$ [o/p variable] at any time $t \geq t_0$ can be computed if we know the applied force $f(t)$ from $t=t_0$ and initial velocity $\{v(t_0)\}$, initial displacement $\{x(t_0)\}$ are known.

State: The state of dynamic system is the minimal set of variables known as state variables, such that the knowledge of these variables at $t=t_0$ together with the knowledge of i/p for $t > t_0$, completely determine the behaviour of the system for $t > t_0$.

State variables: A smallest set of variables which determine the state of the system is called state variable.

State space: It refers to the smallest set of variables in n -dimensional space with ' n ' state variables and it can be determine complete response of the system at any time.

Let us consider a system with m inputs,
 p outputs and n state variables.



Here the state equation gives the relation b/w state variable and inputs :
so the general system can be described by state variables as follows.

[illegible]

The above set of equations can be represented by

$$\dot{x} = f(x(t), u(t))$$

for time variant system, the equation will be

$$\dot{x} = f(\underline{x}(t), \underline{u}(t), t)$$

Similarly the o/p equation is a function of 'Input' and state variables i.e.

$$y(t) = g(\underline{x}(t), \underline{u}(t))$$

state variables

$y(t) = g(\underline{x}(t), \underline{u}(t))$ for time invariant system

$y(t) = g(\underline{x}(t), \underline{u}(t), t)$ for time variant system

In this chapter, we concerned with time invariant system in which the term $\dot{x}$ is a linear combination of state variables and inputs, i.e.

$$\begin{aligned}\dot{x}_1 &= a_{11}x_1(t) + a_{12}x_2(t) + \dots + a_{1n}x_n(t) + b_{11}u_1(t) + b_{12}u_2(t) \\ &\quad + \dots + b_{1m}u_m(t) \\ \dot{x}_2 &= a_{21}x_1(t) + a_{22}x_2(t) + \dots + b_{21}u_1(t) + \dots + b_{2m}u_m(t) \\ &\vdots \\ \dot{x}_n &= a_{n1}x_1(t) + \dots + a_{nn}x_n(t) + b_{n1}u_1(t) + \dots + b_{nm}u_m(t)\end{aligned}$$

The above equations can be represented in matrix form

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \vdots \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{bmatrix} + \begin{bmatrix} b_{11} & b_{12} & \dots & b_{1m} \\ b_{21} & b_{22} & \dots & b_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & \dots & b_{nm} \end{bmatrix} \begin{bmatrix} u_1(t) \\ u_2(t) \\ \vdots \\ u_m(t) \end{bmatrix}$$

$$\boxed{\dot{X} = AX + BU}$$

where X is $n \times 1$ state vector

U is $m \times 1$ input vector

A is $n \times n$ square matrix

B is $n \times m$ matrix

Similarly, the o/p of the system can be expressed as a linear combination of state variables and i/p

$$\begin{aligned} y_1 &= c_{11}x_1 + c_{12}x_2 + \dots + c_{1n}x_n + d_{11}u_1 + \dots + d_{1m}u_m \\ &\vdots \\ y_p &= c_{p1}x_1 + c_{p2}x_2 + \dots + c_{pn}x_n + d_{p1}u_1 + \dots + d_{pm}u_m \end{aligned}$$

It can be represented in matrix form as,

$$\begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_p \end{bmatrix} = \begin{bmatrix} c_{11} & c_{12} & \dots & c_{1n} \\ \vdots & \vdots & & \vdots \\ c_{p1} & c_{p2} & \dots & c_{pn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} + \begin{bmatrix} d_{11} & d_{12} & \dots & d_{1m} \\ \vdots & \vdots & & \vdots \\ d_{p1} & d_{p2} & \dots & d_{pm} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_m \end{bmatrix}$$

$$Y = CX + DU$$

$\swarrow$ $p \times 1$ o/p vector $\downarrow$ $p \times n$ o/p matrix $\searrow$ $p \times m$ matrix

The state models of LTI systems can be represented by following equations

$$\begin{aligned} \dot{X}(t) &= AX + BU \\ Y(t) &= CX + DU \end{aligned}$$

$\rightarrow$ state equation
 $\rightarrow$ o/p equation

State model for differential equation:

→ write the state model for following system which is represented as

$$\dddot{y} + 4\ddot{y} + 6\dot{y} + 8y = 10u$$

Sol: State model is

$$\dot{X} = AX + BU$$

$$Y = CX + DU$$

The order of differential equation = 3

Hence Number of state variables are 3 i.e. $n=3$

Let

$$x_1 = y$$

$$\dot{x}_1 = x_2 = \dot{y} \rightarrow (1)$$

$$\dot{x}_2 = x_3 = \ddot{y} \rightarrow (2)$$

$$\dot{x}_3 = \dddot{y}$$

From given D.E, we can write

$$\dddot{y} = -4\ddot{y} - 6\dot{y} - 8y + 10u$$

$$\dot{x}_3 = -4x_3 - 6x_2 - 8x_1 + 10u \rightarrow (3)$$

$$y = x_1 \rightarrow (4)$$

$$\begin{aligned} \dot{X} &= AX + BU \\ Y &= CX + DU \end{aligned}$$

State model will be

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -8 & -6 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 10 \end{bmatrix} u$$

$3 \times 3 \qquad \qquad \qquad 3 \times 1$

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -8 & -6 & -4 \end{bmatrix}$$

$$B = \begin{bmatrix} 0 \\ 0 \\ 10 \end{bmatrix}$$

$$D = 0$$

$$Y = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \end{bmatrix} u$$

$1 \times 3 \qquad \qquad \qquad 1 \times 1$

Write the state model for the following D.E

$$\frac{d^3 y}{dt^3} + 5 \frac{d^2 y}{dt^2} + 10 \frac{dy}{dt} + 2y = 5u$$

Sol:

The required state model is

$$\dot{X} = AX + BU$$

$$Y = CX + DU$$

Given differential equation

$$\ddot{y} + 5\ddot{y} + 10\dot{y} + 2y = 5u$$

Let

$$\begin{aligned} x_1 &= y \rightarrow \textcircled{1} \\ \dot{x}_1 &= x_2 = \dot{y} \rightarrow \textcircled{2} \\ \dot{x}_2 &= x_3 = \ddot{y} \rightarrow \textcircled{3} \\ \dot{x}_3 &= \ddot{y} \end{aligned}$$

From given D.E

$$\begin{aligned} \ddot{y} &= -5\ddot{y} - 10\dot{y} - 2y + 5u \\ \dot{x}_3 &= -5x_3 - 10x_2 - 2x_1 + 5u \rightarrow \textcircled{4} \end{aligned}$$

Then state model is

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -2 & -10 & -5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 5 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \end{bmatrix} u$$

State model for Transfer function:

→ Derive state model for the following T.F

$$\frac{Y(s)}{U(s)} = \frac{2s+3}{s^2+5s+6}$$

No. of poles = 2, then no. of state variables = 2

Sol:

Let $s^0 = x_1$
 $s = \dot{x}_1 = x_2$ → ①
 $s^2 = \dot{x}_2$

From given T.F, we can write by equating numerators and denominators

$$Y(s) = 2s+3$$

$$Y = 2s+3s^0$$

$$Y = 2x_2 + 3x_1$$
→ ③

$$U(s) = s^2+5s+6$$

$$U = \dot{x}_2 + 5x_2 + 6x_1$$

$$\dot{x}_2 = -6x_1 - 5x_2 + U$$
→ ②

$$\dot{X} = AX + BU$$

$$Y = CX + DU$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -6 & -5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} U$$

$$Y = \begin{bmatrix} 3 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \end{bmatrix} U$$

→ Derive state model of TF $\frac{Y(s)}{U(s)} = \frac{1}{(s+1)(s+2)(s+3)}$

Solⁿ

No. of poles are '3', then no. of state variables = 3

Let $s^0 = x_1$

$s = \dot{x}_1 = x_2 \rightarrow (1)$

$s^2 = \dot{x}_2 = x_3 \rightarrow (2)$

$s^3 = \dot{x}_3$

Equate numerator & denominator

$Y(s) = 1$

$Y(s) = s^0$

$y = x_1 \rightarrow (4)$

$U(s) = (s+1)(s+2)(s+3)$

$U = s^3 + 6s^2 + 11s + 6$

$U = \dot{x}_3 + 6x_3 + 11x_2 + 6x_1$

$\ddot{x}_3 = -6x_1 - 11x_2 - 6x_3 + U$
→ (3)

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} U$$

$$y = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + 0$$

State model for electrical Networks:

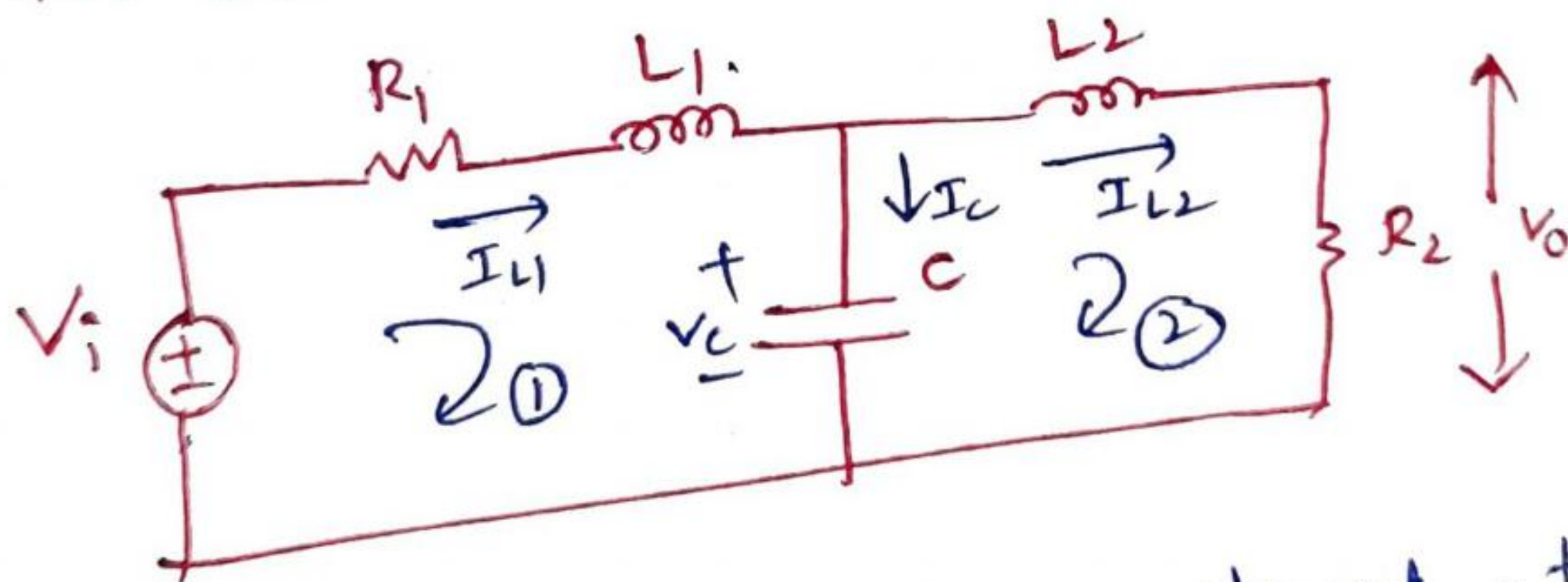
procedure:

step 1) select state variable as voltage across the capacitor, current through inductor. Here, the no. of state variables are equal to the no. of energy storage elements.

step 2) write independent KCL and KVL equations
[apply KCL at capacitor junction,
" KVL to inductor]

step 3) The resultant equation should consist of differential state variable, input and o/p variables.

→ Find the state model for following electrical N/w



Sol:

There are '3' energy storage elements. Hence

No. of state variables are '3'.

The state variables are V_C , I_{L1} , I_{L2}
 $\downarrow$ $\downarrow$ $\downarrow$
 x_1 x_2 x_3

$$\begin{aligned} \dot{x}_1 &= \dot{V}_C \\ \dot{x}_2 &= \dot{I}_{L1} \\ \dot{x}_3 &= \dot{I}_{L2} \end{aligned}$$

Apply KCL at capacitor junction

$$I_{L1} = I_C + I_{L2}$$

$$I_{L1} = I_{L2} + C \cdot \frac{dV_C}{dt}$$

$$C \frac{dV_C}{dt} = I_{L1} - I_{L2}$$

$$\dot{x}_1 = \dot{V}_C = \frac{dV_C}{dt} = \frac{1}{C} I_{L1} - \frac{1}{C} I_{L2} \rightarrow (1)$$

KVL at loop (1)

$$-V_i + I_{L1} R_1 + L_1 \frac{dI_{L1}}{dt} + V_C = 0$$

$$L_1 \frac{dI_{L1}}{dt} = -V_C - I_{L1} R_1 + V_i$$

$$\dot{x}_2 = \dot{I}_{L1} = \frac{dI_{L1}}{dt} = \frac{-V_C}{L_1} - \frac{R_1}{L_1} I_{L1} + \frac{1}{L_1} V_i \rightarrow (2)$$

KVL at loop (2)

$$-V_C + R_2 I_{L2} + L_2 \frac{dI_{L2}}{dt} = 0$$

$$\dot{x}_3 = \dot{I}_{L2} = \frac{dI_{L2}}{dt} = \frac{1}{L_2} V_C - \frac{R_2}{L_2} I_{L2} \rightarrow (3)$$

State model.

$$\dot{x} = Ax + Bu$$

$$y = Cx + Du$$

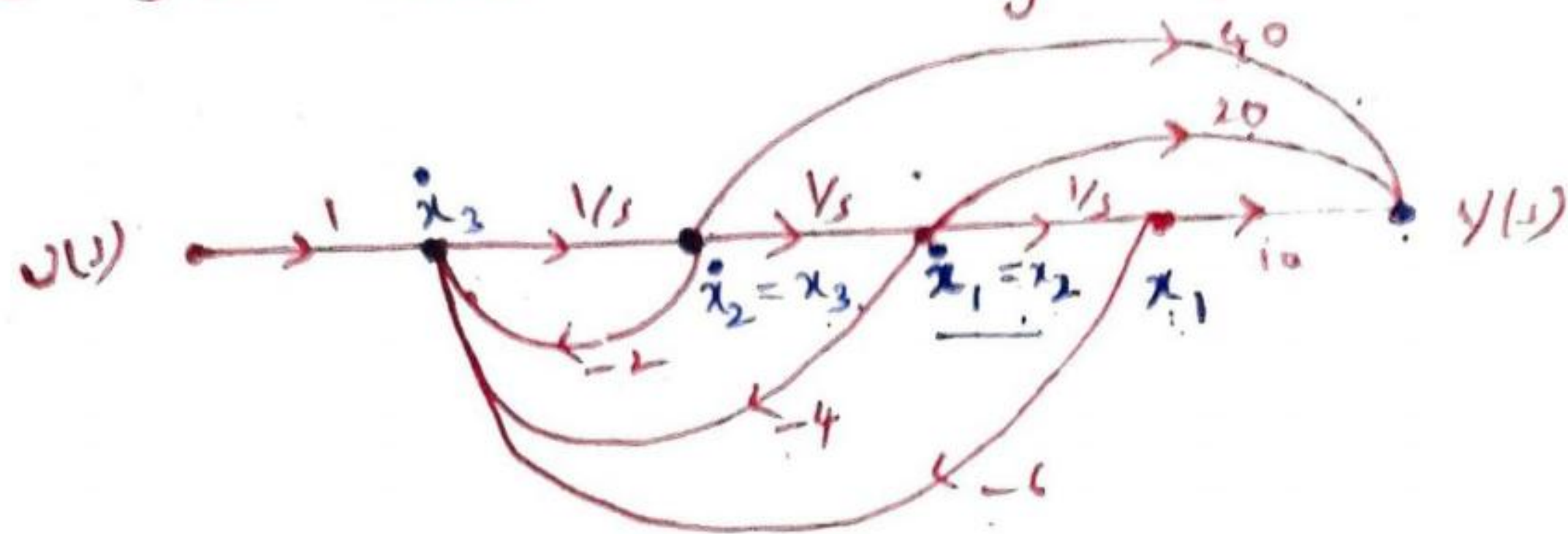
$$\begin{bmatrix} \dot{V}_C \\ \dot{I}_{L1} \\ \dot{I}_{L2} \end{bmatrix} = \begin{bmatrix} 0 & \frac{1}{C} & -\frac{1}{C} \\ -\frac{1}{L_1} & -\frac{R_1}{L_1} & 0 \\ \frac{1}{L_2} & 0 & -\frac{R_2}{L_2} \end{bmatrix} \begin{bmatrix} V_C \\ I_{L1} \\ I_{L2} \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{L_1} \\ 0 \end{bmatrix} V_i \rightarrow (4)$$

$$V_o = I_{L2} R_2$$

$$V_o = \begin{bmatrix} 0 & 0 & R_2 \end{bmatrix} \begin{bmatrix} V_C \\ I_{L1} \\ I_{L2} \end{bmatrix} + 0$$

State model for signal flow graph:

→ Derive state model for following signal flow graph



Sol:

from signal flow graph

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = x_3$$

$$\dot{x}_3 = -6x_1 - 4x_2 - 2x_3 + u(s)$$

$$y = 10x_1 + 20x_2 + 40x_3$$

State model

$$\dot{x} = Ax + Bu$$

$$y = Cx + Du$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -4 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u$$

$$y = \begin{bmatrix} 10 & 20 & 40 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + 0 u$$

Derivation of Transfer function from state model :

Consider state equations

$$\dot{X}(t) = AX(t) + BU(t) \rightarrow (1)$$

$$Y(t) = CX(t) + DU(t) \rightarrow (2)$$

Apply L.T in above equations by considering initial condition

$$(1) \Rightarrow sX(s) - X(0) = AX(s) + BU(s) \rightarrow (3)$$

$$(2) \Rightarrow Y(s) = CX(s) + DU(s) \rightarrow (4)$$

Consider equation (3)

$$sX(s) - X(0) = AX(s) + BU(s)$$

$$sX(s) - AX(s) = X(0) + BU(s)$$

$$X(s)[sI - A] = X(0) + BU(s)$$

$$X(s) = [sI - A]^{-1} X(0) + [sI - A]^{-1} BU(s) \rightarrow (5)$$

Substitute (5) in (4)

$$Y(s) = C \left[[sI - A]^{-1} X(0) + [sI - A]^{-1} BU(s) \right] + DU(s)$$

Assume initial condition $X(0) = 0$, Then

$$Y(s) = [C[sI - A]^{-1}B + D]U(s)$$

$$T.F = \frac{Y(s)}{U(s)} = C[sI - A]^{-1}B + D$$

$$T.F = C \frac{\text{adj}[sI - A]}{|sI - A|} \cdot B + D$$

→ Find Transfer function for the following state model.

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -6 & -5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

$$y = \begin{bmatrix} 3 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Sol:

From given state model.

$$A = \begin{bmatrix} 0 & 1 \\ -6 & -5 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad C = \begin{bmatrix} 3 & 2 \end{bmatrix}, \quad D = 0$$

T.F of a system is given by

$$T.F = C [sI - A]^{-1} B + D$$

$$sI - A = s \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ -6 & -5 \end{bmatrix}$$

$$= \begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ -6 & -5 \end{bmatrix} = \begin{bmatrix} s & -1 \\ 6 & s+5 \end{bmatrix}$$

$$[sI - A]^{-1} = \frac{\text{adj}[sI - A]}{|sI - A|}$$

$$\left. \begin{aligned} A &= \begin{bmatrix} a & b \\ c & d \end{bmatrix} \\ \text{adj } A &= \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \end{aligned} \right\}$$

$$= \frac{\begin{bmatrix} s+5 & 1 \\ -6 & s \end{bmatrix}}{s(s+5) - (-6)}$$

$$[sI - A]^{-1} = \frac{1}{s^2 + 5s + 6} \begin{bmatrix} s+5 & 1 \\ -6 & s \end{bmatrix}$$

$$T.F = C [sI - A]^{-1} B + D$$

$$= \frac{1}{s^2 + 5s + 6} \begin{bmatrix} 3 & 2 \end{bmatrix}_{1 \times 2} \begin{bmatrix} s+5 & 1 \\ -6 & s \end{bmatrix}_{2 \times 2} \begin{bmatrix} 0 \\ 1 \end{bmatrix}_{2 \times 1}$$

$$= \frac{1}{s^2 + 5s + 6} \begin{bmatrix} 3 & 2 \end{bmatrix}_{1 \times 2} \begin{bmatrix} 1 \\ s \end{bmatrix}_{2 \times 1}$$

$$T.F = \frac{Y(s)}{U(s)} = \frac{2s+3}{s^2 + 5s + 6}$$

$$\text{poles of T.F } s^2 + 5s + 6 = 0$$

$$s = -2, -3$$

poles are lying on left half of s-plane

system is stable system.

$|sI - A| = 0$ will give poles of CTTF.

→ Find T.F for the following state model.

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \underbrace{\begin{bmatrix} -2 & -3 \\ 4 & 2 \end{bmatrix}}_A \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \underbrace{\begin{bmatrix} 3 \\ 5 \end{bmatrix}}_B u$$

$$y = \underbrace{\begin{bmatrix} 1 & 1 \end{bmatrix}}_C \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Sol:

$$sI - A = \begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \begin{bmatrix} -2 & -3 \\ 4 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} s+2 & +3 \\ -4 & s-2 \end{bmatrix}$$

$$[sI - A]^{-1} = \frac{\text{adj}(sI - A)}{|sI - A|}$$

$$= \frac{1}{s^2 - 4 + 12} \begin{bmatrix} s-2 & -3 \\ 4 & s+2 \end{bmatrix}$$

$$T.F = C [sI - A]^{-1} B + D$$

$$= \frac{1}{s^2 + 8} \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} s-2 & -3 \\ 4 & s+2 \end{bmatrix} \begin{bmatrix} 3 \\ 5 \end{bmatrix}$$

$$= \frac{1}{s^2 + 8} \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} 3s - 21 \\ 5s + 22 \end{bmatrix}$$

$$= \frac{1}{s^2 + 8} [3s - 21 + 5s + 22]$$

$$\boxed{T.F = \frac{8s + 1}{s^2 + 8}}$$

Solution of state equations.

consider state equation.

$$\dot{X} = AX(t) + BU(t) \rightarrow (1)$$

$$Y = CX(t) + DU(t) \rightarrow (2)$$

In equation (1)

Case (i) If matrix A is constant and i/p vector is zero i.e. no control force is applied on the system, then the system is homogeneous system.

$$\dot{X} = AX(t)$$

It has only zero input response (ZIR) is solution

Case (ii) If matrix A is constant and input vector is non zero then system is called non homogeneous system.

$$\dot{X} = AX(t) + BU(t)$$

It has ZIR, and ZSR (zero state response) is solution.

Solution of Non homogeneous state equations:

Consider a Non homogeneous state equations

$$\dot{X}(t) = AX(t) + Bu(t) \rightarrow (1)$$

and assume initial condition $X(0) = X_0$

$$\dot{X}(t) - AX(t) = Bu(t)$$

Multiply above equation with e^{-At}

$$e^{-At} \cdot \dot{X}(t) - e^{-At} \cdot AX(t) = Bu(t) \cdot e^{-At}$$

$$e^{-At} \cdot \frac{dX(t)}{dt} + X(t) \cdot \frac{d}{dt}(e^{-At}) = Bu(t) \cdot e^{-At}$$

$$\frac{d}{dt} [e^{-At} X(t)] = Bu(t) \cdot e^{-At}$$

Integrate above equation w.r.t to 't'

$$e^{-At} X(t) \Big|_0^t = \int_0^t Bu(\tau) \cdot e^{-A\tau} d\tau$$

$$e^{-At} X(t) - X(0) = \int_0^t Bu(\tau) \cdot e^{-A\tau} d\tau$$

$$e^{-At} X(t) = X(0) + \int_0^t Bu(\tau) \cdot e^{-A\tau} d\tau$$

$$X(t) = \underbrace{e^{At} X(0)}_{ZIR} + \underbrace{\int_0^t B \cdot u(\tau) \cdot e^{-A(\tau-t)} d\tau}_{ZSR}$$

$$\boxed{\phi(t) = e^{At}}$$

$\phi(t)$ is called state transition matrix (STM)

State Transition Matrix (STM) $\phi(t)$

From above equation it is observed that the initial state $x(0)$ or x_0 at $t=0$ is given to state $x(t)$ at time t , so this transition and state is carried out by the matrix e^{At} , because of this property e^{At} is known as state transition matrix.

It is denoted by

$$\boxed{\phi(t) = e^{At}}$$

Laplace Transform solution for state model:-

Consider Non homogeneous state equation

$$\dot{x}(t) = Ax(t) + Bu(t) \rightarrow \textcircled{1}$$

Assume initial condition at $t=0$, $x(0) = x_0$

Apply Laplace Transform for above equation

$$sX(s) - x(0) = AX(s) + BU(s)$$

$$sX(s) - AX(s) = x(0) + BU(s)$$

$$X(s)[sI - A] = x(0) + BU(s)$$

$$\hat{X}(s) = [sI - A]^{-1} x(0) + [sI - A]^{-1} BU(s)$$

Apply Inverse L.T

$$X(t) = \mathcal{L}^{-1} \left[(sI - A)^{-1} X(0) + (sI - A)^{-1} B U(s) \right]$$

$$X(t) = \underbrace{\mathcal{L}^{-1} [(sI - A)^{-1} X(0)]}_{\text{ZIR}} + \underbrace{\mathcal{L}^{-1} [(sI - A)^{-1} B U(s)]}_{\text{ZSR} \rightarrow \text{X(2)}}$$

By comparing both the solutions, we can write

$$\text{STM} \quad \phi(t) = e^{At} = \mathcal{L}^{-1} [(sI - A)^{-1}]$$

Properties of state Transition matrix:

property 1: $\phi(0)$ is an identity matrix i.e. $\phi(0) = I$

proof: Consider solution of homogeneous state eqn

$$X(t) = e^{At} X(0)$$

$$X(t) = \phi(t) X(0)$$

At $t=0$

$$X(0) = \phi(0) X(0)$$

It is in the form $\uparrow$

$$A = AI = IA$$

From the above eqn, $\phi(t)$ is an identity matrix

property ②

$$\phi(-t) = \phi(t)^{-1}$$

proof:

we know

$$\phi(t) = e^{At}$$

$$t = -t$$

$$\phi(-t) = e^{A(-t)}$$

$$\phi(-t) = (e^{At})^{-1}$$

$$\boxed{\phi(t) = \phi(t)^{-1}}$$

property ①

$$\phi(t+t_2) = \phi(t_1) \cdot \phi(t_2)$$

proof

$$\phi(t) = e^{At}$$

$$t = t_1 + t_2$$

$$\phi(t+t_2) = e^{A(t_1+t_2)}$$

$$= e^{At_1 + At_2}$$

$$= e^{At_1} \cdot e^{At_2}$$

$$\phi(t+t_2) = \phi(t_1) \phi(t_2)$$

property ④

$$\phi^m(t) = \phi(mt)$$

proof

$$\phi(t) = e^{At}$$

$$\phi^m(t) = (e^{At})^m$$

$$= e^{A(mt)}$$

$$\phi^m(t) = \phi(mt)$$

property ⑤

$$\phi(t_2 - t_1) \phi(t_1 - t_0) = \phi(t_2 - t_0)$$

proof

$$\phi(t_2 - t_1) \phi(t_1 - t_0) = e^{A(t_2 - t_1)} \cdot e^{A(t_1 - t_0)}$$

$$= e^{At_2 - At_1} \cdot e^{At_1 - At_0}$$

$$= e^{At_2 - At_1 + At_1 - At_0}$$

$$= e^{At_2 - At_0}$$

$$= e^{A(t_2 - t_0)}$$

$$\boxed{\phi(t_2 - t_1) \phi(t_1 - t_0) = \phi(t_2 - t_0)}$$

Obtain state transition matrix if

$$A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$$

Solⁿ

STM is given by

$$e^{At} = \phi(t) = L^{-1} \{ (sI - A)^{-1} \}$$

$$sI - A = \begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} s-1 & 0 \\ -1 & s-1 \end{bmatrix}$$

$$[sI - A]^{-1} = \frac{\text{adj}(sI - A)}{(sI - A)}$$

$$= \frac{1}{(s-1)^2 + 0} \begin{bmatrix} s-1 & 0 \\ 1 & s-1 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{s-1} & 0 \\ \frac{1}{(s-1)^2} & \frac{1}{s-1} \end{bmatrix}$$

Apply I. L. T

$$L^{-1} \{ (sI - A)^{-1} \} = \begin{bmatrix} L^{-1} \left\{ \frac{1}{s-1} \right\} & 0 \\ L^{-1} \left\{ \frac{1}{(s-1)^2} \right\} & L^{-1} \left\{ \frac{1}{s-1} \right\} \end{bmatrix}$$

$$\phi(t) = \begin{bmatrix} e^t & 0 \\ t \cdot e^t & e^t \end{bmatrix}$$

→ The LTI system is characterized by a state equation, where 'u' is unit step function. and

$$x(0) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 2 \end{bmatrix} u$$

Det. solution of above state equation.

Solution is given by

$$x(t) = ZIR + ZSR$$

$$x(t) = \mathcal{L}^{-1} \left[(sI - A)^{-1} \right] x(0) + \mathcal{L}^{-1} \left[(sI - A)^{-1} B U(s) \right]$$

① ZIR

$$sI - A = \begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}$$

$$= \begin{bmatrix} s & -1 \\ +2 & s+3 \end{bmatrix}$$

$$(sI - A)^{-1} = \frac{1}{s(s+3)+2} \begin{bmatrix} s+3 & 1 \\ -2 & s \end{bmatrix}$$

$$= \frac{1}{s^2+3s+2} \begin{bmatrix} s+3 & 1 \\ -2 & s \end{bmatrix}$$

$$(sI - A)^{-1} = \begin{bmatrix} \frac{s+3}{(s+1)(s+2)} & \frac{1}{(s+1)(s+2)} \\ \frac{-2}{(s+1)(s+2)} & \frac{s}{(s+1)(s+2)} \end{bmatrix}$$

From partial fraction

$$\frac{s+3}{(s+1)(s+2)} = \frac{A}{s+1} + \frac{B}{s+2}$$

$$s+3 = A(s+2) + B(s+1)$$

$$\text{when } s = -2 \Rightarrow B = -1$$

$$s = -1 \Rightarrow A = 2$$

$$\frac{1}{(s+1)(s+2)} = \frac{1}{s+1} - \frac{1}{s+2}$$

$$\frac{-2}{(s+1)(s+2)} = \frac{-2}{s+1} + \frac{2}{s+2}$$

$$\frac{s}{(s+1)(s+2)} = \frac{-1}{s+1} + \frac{2}{s+2}$$

$$(sI - A)^{-1} = \begin{bmatrix} \frac{2}{s+1} - \frac{1}{s+2} & \frac{1}{s+1} - \frac{1}{s+2} \\ \frac{-2}{s+1} + \frac{2}{s+2} & \frac{-1}{s+1} + \frac{2}{s+2} \end{bmatrix}$$

Applying I.L.T

$$\mathcal{L}^{-1}[(sI - A)^{-1}] = \begin{bmatrix} 2e^{-t} - e^{-2t} & e^{-t} - e^{-2t} \\ -2e^{-t} + 2e^{-2t} & -e^{-t} + 2e^{-2t} \end{bmatrix}$$

$$ZIR = \mathcal{L}^{-1}[(sI - A)^{-1}] \times (0)$$

$$\begin{bmatrix} 2e^{-t} - e^{-2t} & e^{-t} - e^{-2t} \\ -2e^{-t} + 2e^{-2t} & -e^{-t} + 2e^{-2t} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$ZIR = \begin{bmatrix} e^{-t} - e^{-2t} \\ -e^{-t} + 2e^{-2t} \end{bmatrix}$$

② ZSR:

$$z_{SR} = \mathcal{L}^{-1} \left[(sI - A)^{-1} B U(s) \right]$$

$$(sI - A)^{-1} B U(s) = \begin{bmatrix} \frac{s+1}{(s+1)(s+2)} & \frac{1}{(s+1)(s+2)} \\ \frac{-2}{(s+1)(s+2)} & \frac{s}{(s+1)(s+2)} \end{bmatrix} \begin{bmatrix} 0 \\ 2 \end{bmatrix} \frac{1}{s}$$

$$= \begin{bmatrix} \frac{1}{(s+1)(s+2)} \\ \frac{2s}{(s+1)(s+2)} \end{bmatrix} \frac{1}{s}$$

$$= \begin{bmatrix} \frac{1}{s(s+1)(s+2)} \\ \frac{2}{(s+1)(s+2)} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{s} - \frac{2}{s+1} + \frac{1}{s+2} \\ \frac{2}{s+1} - \frac{2}{s+2} \end{bmatrix}$$

$$\mathcal{L}^{-1} \left[(sI - A)^{-1} B U(s) \right] = \begin{bmatrix} 1 - 2e^{-t} + e^{-2t} \\ 2e^{-t} - 2e^{-2t} \end{bmatrix}$$

Solution $x(t) = z_{IR} + z_{SR}$

$$= \begin{bmatrix} e^{-t} - e^{-2t} \\ -e^{-t} + 2e^{-2t} \end{bmatrix} + \begin{bmatrix} 1 - 2e^{-t} + e^{-2t} \\ 2e^{-t} - 2e^{-2t} \end{bmatrix}$$

$$x(t) = \begin{bmatrix} 1 - e^{-t} \\ e^{-t} \end{bmatrix}$$

Q6

Controllability and observability:

Controllability: A system is said to be completely state controllable if it is possible to take the system state from any initial state $x(t_0)$ to any desired state $x(t)$, in specified time by control vector $u(t)$.

Kalman test for Controllability

The mathematic test for controllability is developed by Kalman. The test of controllability due to Kalman which can be applied to any state model, is stated below.

A general n^{th} order multiple i/p multiple o/p LTI system

$$\dot{x} = Ax + Bu$$

is completely controllable if and only if the rank of composite matrix

$$Q_c = [B \quad AB \quad A^2B \quad \dots \quad A^{n-1}B]$$

is equal to n , and $|Q_c| \neq 0$

Observability:

A system is said to be completely observable, ~~implying that~~ if every state $x(t_0)$ can be completely identified by measurements of the output $y(t)$ over finite time interval.

Kalman test for observability:

Consider the state model of an n^{th} order

LTI system

$$\dot{x} = Ax + Bu$$

$$y = Cx + Du$$

According to Kalman's test, a system is completely observable if and only if the composite matrix

$$Q_0 = \begin{bmatrix} C^T & A^T C^T & \dots & (A^T)^{n-1} C^T \end{bmatrix}$$

Rank is equal to ' n ' and $|Q_0| \neq 0$

Q_0 can be also defined as

$$Q_0 = \begin{bmatrix} C \\ CA \\ CA^2 \\ \vdots \\ CA^{n-1} \end{bmatrix}$$

Check whether the system is represented by

$$\dot{X}(t) = \begin{bmatrix} 0 & 5 \\ -1 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u(t)$$

$$y(t) = \begin{bmatrix} 1 & 7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

is controllable/observable
A not

Sol:

By given system

$$A = \begin{bmatrix} 0 & 5 \\ -1 & -2 \end{bmatrix}_{2 \times 2}, B = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, C = \begin{bmatrix} 1 & 7 \end{bmatrix}$$

check for controllability

composite matrix is given by

$$Q_c = [B \quad AB \quad A^2B \quad \dots \quad A^{n-1}B]$$

$$\text{But } n = 2$$

$$Q_c = [B \quad AB]$$

$$AB = \begin{bmatrix} 0 & 5 \\ -1 & -2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 5 \\ -2 \end{bmatrix}$$

$$Q_c = \begin{bmatrix} \overset{\rightarrow B}{1} & \overset{\rightarrow AB}{5} \\ 1 & -2 \end{bmatrix}$$

$$|Q_c| = -2 - 5 = -7 \neq 0$$

$$\text{Rank of } Q_c = 2 = n$$

then the system said to be completely controllable

check for observability

$$Q_o = \begin{bmatrix} C^T & A^T C^T & \dots & (A^T)^{n-1} C^T \end{bmatrix}$$

$$Q_o = \begin{bmatrix} C \\ CA \\ CA^2 \\ \vdots \\ CA^{n-1} \end{bmatrix}$$

Here $n = 2$

$$Q_o = \begin{bmatrix} C \\ CA \end{bmatrix}$$

$$CA = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 0 & 5 \\ -1 & -2 \end{bmatrix}$$
$$= \begin{bmatrix} -1 & +3 \end{bmatrix}$$

$$Q_o = \begin{bmatrix} 1 & 1 \\ -1 & +3 \end{bmatrix}$$

$$|Q_o| = 3 + 1 = 4 \neq 0$$

$$\text{Rank of } Q_o = 2 = n$$

system is completely observable.

$$x(t) = \begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u(t)$$

$$y(t) = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

check whether

of controllable / observable and

Soln:

$$A = \begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix}$$

$$B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$C = \begin{bmatrix} 1 & 0 \end{bmatrix}$$

controllability

$$Q_c = \begin{bmatrix} B & AB & \dots & A^{n-1}B \end{bmatrix}$$

but $n=2$

$$Q_c = \begin{bmatrix} B & AB \end{bmatrix}$$

$$AB = \begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$Q_c = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}$$

$$|Q_c| = 2 - 0 = 2 \neq 0$$

$$\text{Rank} = 2 \Rightarrow$$

completely controllable.

Observability

$$Q_o = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix}$$

$$= \begin{bmatrix} C \\ CA \end{bmatrix}$$

$$CA = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix}$$

$$CA = \begin{bmatrix} 1 & -1 \end{bmatrix}$$

$$Q_o = \begin{bmatrix} 1 & 0 \\ 1 & -1 \end{bmatrix}$$

$$|Q_o| = 1(-1) - 0 = -1 \neq 0$$

$$\text{Rank} = 2$$

System is completely observable.